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October 29, 1982

Mr. A. Schwencer, Chief
Licensing Branch #2
Division of Licensing
U. S. Nuclear Regulatory Commission
Washington, DC 20555

Subject: LaSalle County Station Units 1 and 2
Concerns Regarding the Adequacy of
Design Margins of the Mark I and II
Containment Systems
NRC Docket Nos. 50-373 and 50-374

References (a): R. L. Tedesco letter to L. O. DelGeorge dated July 2, 1982.

(b): C. W. Schroeder letter to A. Schwencer dated August 30, 1982.

Dear Mr. Schwencer:

Reference (a) listed 22 concerns which Mr. John Humphrey had identified regarding Mark III containments. It requested a response to those concerns which were identified as being potentially applicable to LaSalle County Station. Reference (b) provided Commonwealth Edison Company's response to those concerns. On October 4 and 19, 1982, Commonwealth Edison representatives discussed NRC questions on Reference (b) by telecon with Messrs. A. Bournia and Farouk Eltawila.

The purpose of this letter is to provide you with the enclosed written documentation of the discussions. It is our understanding that the NRC concerns were satisfied and that providing this documentation would close this issue for LaSalle County Station.

Enclosed for your use are one (1) signed original and thirty-nine (39) copies of this letter and the enclosure.

If there are any further questions in this matter, please contact this office.

Very truly yours,

CW Schroeder 10/29/82

C. W. Schroeder
Nuclear Licensing Administrator

lm

Enclosure

cc: NRC Resident Inspector - LSCS

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LASALLE RESPONSES TO NRC QUESTIONS
ON THE HUMPHREY CONCERNS

Question 3.1 - What load combinations were considered with RHR relief valve discharge? What is the reference to the condensation map in DFFR?

Response - The RHR steam condensing mode is a secondary method of reactor cooldown not normally used in plant operations. By virtue of the time required to align RHR into the steam condensing mode (greater than 20 min.), it is not expected that the reactor would be maximum pressure (conditions for highest loads) nor in a transient when SRV's are needed to control pressure while RHR is in the steam condensing mode.

The postulated combination of events, ie. RHR in the steam condensing mode, reactor at > 500 psig, pressure controller fails high, plus LOCA was judged too improbable to include RHR relief valve loads in the DFFR load combinations. Further, the SRV loads bound RHR loads by about 50%, so LOCA plus SRV combinations bound LOCA plus RHR.

Question 3.3 - Provide justification for $f \propto m^{-1/3}$.

Response - The justification for the formula for frequency is given in the following reference (copies attached):

1. Devin, Jr., Charles, "Survey of Thermal, Radiation, and Viscous Damping of Pulsating Air Bubbles in Water", The Journal of the Acoustical Society of America, Volume 31, Number 12, December, 1959, pages 1654-1667.

2. Foody, B.E., Huber, P.W., "On the Radial Oscillations of Multiple Gas Bubbles in an Incompressible Liquid", Journal of Applied Mechanics, Volume 48, December, 1981, pages 727-731.

$$f = \frac{1}{2\pi R_o} \left(\frac{3\gamma g_c P_o}{\rho} \right)^{1/2}$$

Reference 1
Equation 2-1

$$V_o = \frac{4}{3} \pi R_o^3$$

$$V_o = \frac{m \; R T_o}{P_o}$$

$$R_o = \frac{3}{4\pi} \left(\frac{m \; R \; T_o}{P_o} \right)^{1/3}$$

$$f = \frac{1}{2\pi} \left(\frac{4\pi P_o}{3 \; m \; R \; T_o} \right)^{1/3} \left(\frac{3\gamma g_c \; P_o}{\rho} \right)^{1/2}$$

$$f = \frac{1}{2\pi} \frac{P_o^{5/6}}{m^{1/3}} \left(\frac{4\pi}{3\ RT_o}\right)^{1/3} \left(\frac{3\gamma g_c}{\rho}\right)^{1/2}$$

$$c = \frac{1}{2\pi} \left(\frac{4\pi}{3\ RT_o}\right)^{1/3} \left(\frac{3\gamma g_c}{\rho}\right)^{1/2}$$

$$f = C \frac{P_o^{5/6}}{m^{1/3}}$$

$$f \sim P_o^{5/6}$$

$$f \sim m^{-1/3}$$

Question 3.6 - When the RHR heat exchanger is in the steam condensing mode, what happen s when a LOCA occurs?

Response - When a LOCA occurs, the valving in the RHR system automatically realigns itself to the LPCI injection mode. The action is taken on either low RPV water level or high drywell pressure signal. Within the first minute of the accident, all of the valves would be realigned.

Question 4.9 - Are there instances when the drywell spray and pool cooling modes are used concurrently?

Response - It is functionally possible to run one independent RHR loop in pool cooling and another in containment spray if conditions warranted. An RHR loop, however, cannot run in the two modes simultaneously. System pressure in the pool cooling mode will not allow water to flow up and out the drywell spray ring header.

Question 5.2 - Is there any interlock between the drywell spray and recombiner?

Response - There are no interlocks between the drywell spray and control recombiner.

Question 6.4 - In certain conditions, the drywell may reach 340°F. What happens to the air monitoring system?

Response - The analyzers are kept at 300°F. At design conditions of 45 psig and 340°F, the steam is superheated. At 45 psig the saturation temperature is equal to 292°F. This temperature is below the temperature at which the analyzers are maintained. It is anticipated that no condensation will take place of the superheated steam, only cooling.

Question 9.2 - Last sentence is unclear.

Response - Last sentence should read: "The RHR wetwell spray mode can mitigate the effects of prolonged leakage into the wetwell having no effect on suppression pool cooling."

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On the Radial Oscillations of Multiple Gas Bubbles in an Incompressible Liquid

The radial oscillations of multiple gas bubbles in an incompressible liquid bounded by plane solid or free surfaces are analyzed. Derivation of the normal mode frequencies and mode shapes of two, three, and four bubble configurations in an unbounded liquid illustrate the method of analysis. Expressions for the oscillation frequencies of bubbles near free and solid surfaces are derived.

Introduction

The dynamics of a single spherical bubble oscillating radially in a liquid have been extensively investigated. Rayleigh [1] derived the basic equation governing the motion of a gas cavity in an unbounded fluid. Minnaert [2], using a simple energy method, determined the natural frequency of an oscillating gas bubble. Since that time, research on vapor bubbles and cavitation has motivated studies of vapor bubble collapse, heat transfer effects, and a wide range of other topics. Hsieh [3, 4] reviews some of this work.

No previous analysis, however, has dealt generally with the hydrodynamic relations governing interactions among multiple oscillating bubbles and their surroundings. It is well known that a solid surface attracts an oscillating bubble while a free surface repels it [5]. Cole [6] and Friedman [7] have derived expressions for the oscillation period of a single, explosively formed bubble situated between a free and solid surface. Since surfaces can be interpreted as the planes of symmetry in a multiple bubble system, these expressions also determine oscillation periods for a restricted set of multiple bubble configurations.

This paper presents a method for estimating the radial oscillation natural frequencies of any number and configuration of gas bubbles in an unbounded liquid or in a liquid bounded by plane solid or free surfaces. The analysis is limited to linear, small amplitude oscillations.

Model Assumptions

We consider the radial oscillations of N gas bubbles in an unbounded liquid. The liquid flow is assumed to be incompressible and irrotational. Gravitational and surface tension effects are ignored. Far from the bubbles, the pressure P_∞ is uniform.

Radial displacements of the bubble surfaces are taken to be small compared to bubble radii r

$$\delta r/r \ll 1. \tag{1}$$

If the velocity in the fluid varies with a period τ and is characterized by an amplitude U_0 on the bubble surface, then δr is of order $U_0\tau$. The Strouhal number, $r/U_0\tau$ is then much greater than 1 and convective accelerations in the liquid can be ignored. Equation (1) also implies that the time scale for oscillation is much smaller than that for convective distortions of the bubble geometry. We shall assume that the bubble geometry is invariant (spherical or ellipsoidal) during the oscillations analyzed. The gas is treated as ideal and adiabatic with a ratio of specific heats γ. Equation (1) can then be written

$$\delta P/3\gamma P_\infty \ll 1 \tag{2}$$

where δP is the maximum change in bubble pressure from the equilibrium.

Analysis

The kinetic energy T of a liquid bounded internally by surfaces A can be expressed as

$$T = -\tfrac{1}{2}\rho \int_A \phi \frac{\partial \phi}{\partial n}\, dA \tag{3}$$

where ϕ is the velocity potential and ρ is the liquid density. The normal vector n is defined as pointing *into* the liquid (i.e., outward from the gas bubbles). In our analysis, the integral is taken over the bubble surfaces.

The rate of change of the ith bubble's volume is given by

$$q_i = \int_{A_i} \frac{\partial \phi}{\partial n}\, dA. \tag{4}$$

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where the integral is taken over the surface of the ith bubble. We define two potentials ϕ_1 and ϕ_2, whose sum is ϕ, such that ϕ_1 is uniform over each bubble surface,

$$\int_{A_i} \frac{\partial \phi_1}{\partial n} dA_i = q_i \quad \text{for} \quad i = 1, 2, \ldots N \tag{5a}$$

and

$$\int_{A_i} \frac{\partial \phi_2}{\partial n} dA_i = 0 \quad \text{for} \quad i = 1, 2, \ldots N. \tag{5b}$$

Physically, ϕ_2 is the amount by which the actual potential ϕ departs from the average potential ϕ, at any point on a bubble's surface. Except in the case of a single bubble in an unconfined liquid, $\partial\phi/\partial n$ will not be uniform over each bubble and thus $\partial\phi_2/\partial n$ may vary over each bubble.

Using the relation

$$\int_A \phi_1 \frac{\partial \phi_2}{\partial n} dA = \int_A \phi_2 \frac{\partial \phi_1}{\partial n} dA \tag{6}$$

we obtain

$$T = \tfrac{1}{2}\rho \left[\int_A \phi_1 \frac{\partial \phi_1}{\partial n} dA + \int_A \phi_2 \frac{\partial \phi_2}{\partial n} dA \right]. \tag{7}$$

The kinetic energy can therefore be defined as the sum of two terms: T_1 which depends on ϕ_1 only, and T_2 which depends on ϕ_2.

Euler's equation for highly unsteady motion,

$$-\nabla P = \rho \nabla(\partial \phi/\partial t), \tag{8}$$

requires that $\partial\phi/\partial t$ be uniform on the bubble surface since the pressure P in the liquid is uniform on that surface. Any initial nonuniformity in ϕ will thus persist through a number of oscillations. As a result, ϕ_2 and therefore T_2 will remain roughly constant over the time scales (several bubble oscillation periods) of interest in this analysis.

We note here that, as a consequence of equation (7), the kinetic energy of any bubble system with given volume rates of expansion is a minimum when ϕ is uniform over each bubble surface.

Since ϕ_1 is uniform on each bubble

$$T = -\tfrac{1}{2}\rho \mathbf{q}^T \boldsymbol{\phi}_1 + T_2 \tag{9}$$

where $\mathbf{q}$ is a column vector of the q_i and $\boldsymbol{\phi}_1$ is a column vector specifying the uniform potential ϕ_1 on each surface. Each q_i is related to $\boldsymbol{\phi}_1$ by Laplace's equation and hence varies linearly with each term of $\boldsymbol{\phi}_1$. The relation can therefore be expressed as

$$\mathbf{q} = \mathbf{C}\boldsymbol{\phi}_1 \tag{10}$$

where $\mathbf{C}$ is a square matrix depending only on geometry. $\mathbf{C}$ is analogous to the capacitance matrix in an electrostatic problem with charged conductors.

The kinetic energy may now be written

$$T = \tfrac{1}{2}\mathbf{q}^T \mathbf{M} \mathbf{q} + T_2 \tag{11}$$

where

$$\mathbf{M} \equiv -\rho \mathbf{C}^{-1}. \tag{12}$$

The potential energy of the ith bubble V_i^* is

$$V_i^* = -\int_{V_{0i}}^{V_{0i}+\delta V_i} (P_i - P_\infty) dV \tag{13}$$

where P_i is the pressure in the ith bubble, V_{0i} is its equilibrium volume and δV_i is the change in volume from V_{0i}. Assuming reversible adiabatic expansion of each gas bubble the total potential energy V^* of the multiple bubble system can be found from equation (13) and expressed as

$$V^* = \tfrac{1}{2}\delta\mathbf{V}^T \mathbf{K} \delta\mathbf{V} \tag{14}$$

where $\delta\mathbf{V}$ is a vector of δV_i and is a diagonal matrix whose elements are

$$K_{ii} = \frac{P_\infty \gamma}{V_{0i}}. \tag{15}$$

Lagrange's equations, with $\delta \dot{V}_i$ substituted for q_i require:

$$\frac{d}{dt}\left(\frac{\partial T}{\partial(\delta \dot{V}_i)}\right) - \frac{\partial T}{\partial(\delta V_i)} = -\frac{\partial V^*}{\partial(\delta V_i)} \quad \text{for} \quad i = 1, 2, \ldots N. \tag{16}$$

If $\mathbf{M}$ does not vary with time, we obtain

$$\mathbf{M}\delta\ddot{\mathbf{V}} + \mathbf{K}\delta\mathbf{V} = 0. \tag{17}$$

The constant term T_2 in the kinetic energy disappears in the differentiation. The eigenvalues of equation (17),

$$|-\omega^2\mathbf{M} + \mathbf{K}| = 0, \tag{18}$$

are the natural frequencies ω of the N bubble system and the eigenvectors define the mode shapes. If the eigenvectors are known *a priori*, ω can be deduced from the Rayleigh-Ritz equation (8)

$$\omega = \sqrt{\frac{\tfrac{1}{2}\delta\mathbf{V}^T\mathbf{K}\delta\mathbf{V}}{\tfrac{1}{2}\delta\mathbf{V}^T\mathbf{M}\delta\mathbf{V}}} \tag{19}$$

where $\delta\mathbf{V}$ is an eigenvector. This formulation is particularly useful when the mode shapes can be deduced directly from considerations of symmetry.

The Mass Matrix. The mass matrix, equation (12), relates ϕ and q. For a single spherical bubble of radius r_0

$$\phi = -\frac{q}{4\pi r_0} \tag{20}$$

and equations (10) and (12) yield

$$M = \frac{\rho}{4\pi r_0}. \tag{21}$$

Substituting equations (21) and (15) into equation (19) yields the well-known Rayleigh bubble natural frequency (2)

$$\omega = \frac{1}{r_0}\sqrt{\frac{3P_\infty\gamma}{\rho}}. \tag{22}$$

For a single ellipsoidal bubble with semiaxes a, b, c, Smythe [9] gives

$$\phi = -\frac{q}{8\pi}\int_0^\infty [(a^2+\theta)(b^2+\theta)(c^2+\theta)]^{-1/2} d\theta. \tag{23}$$

Therefore

$$M = \frac{\rho}{8\pi}\int_0^\infty [(a^2+\theta)(b^2+\theta)(c^2+\theta)]^{-1/2} d\theta. \tag{24}$$

If the ellipsoid is prolate, equation (24) with $a = b$ can be integrated to give

$$M = \frac{\rho}{4\pi a}\left[\frac{-1}{\sqrt{(c/a)^2 - 1}} \log\left[(c/a) - \left[\left(\frac{c}{a}\right)^2 - 1\right]^{1/2}\right]\right] \quad c > a \tag{25a}$$

$$M = \frac{\rho}{4\pi a}\left[\frac{1}{\sqrt{1 - \left(\frac{c}{a}\right)^2}}\left[\frac{\pi}{2} - \tan^{-1}\frac{1}{\sqrt{\left(\frac{a}{c}\right)^2 - 1}}\right]\right] \quad c < a, \tag{25b}$$

Fig. 1 compares bubble frequencies of single spherical and ellipsoidal bubbles of equal volume calculated using equation (19) with equation (21) or (25), respectively. The dependence of bubble frequency on bubble geometry is found to be very weak.

It is more difficult to determine the elements of $\mathbf{M}$ analytically for configurations involving two or more bubbles. Smythe [9] uses a method of images approach [10] to solve the analogous electrostatic problem: the potential energy of two charged conducting spheres of radius r_0 separated center to center by r_{12}. The solution can be expressed in terms of $\mathbf{M}$ as

$$M_{11} = M_{22} = \frac{\rho}{4\pi r_0}\left(\frac{A}{A^2 - B^2}\right) \tag{26a}$$

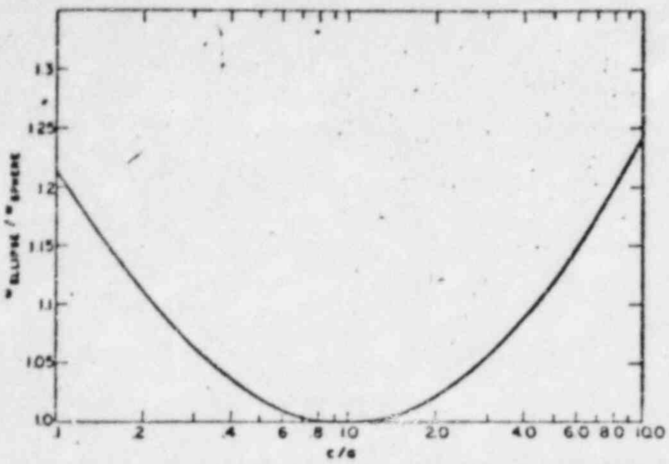

Fig. 1 Ellipsoidal bubble frequency ($\omega_{ellipse}$) compared to the frequency of a spherical bubble of the same volume; ellipsoid semiaxes are c, a, a

$$M_{12} = M_{21} = \frac{\rho}{4\pi r_{12}}\left(\frac{r_{12}}{r_0}\right)\left(\frac{B}{A^2 - B^2}\right) \tag{26b}$$

where

$$A = \sinh\beta \sum_{n=1}^{\infty} \operatorname{csch}[(2n-1)\beta]$$

$$B = \sinh\beta \sum_{n=1}^{\infty} \operatorname{csch}[(2n)\beta]$$

$$\cosh\beta = \frac{r_{12}}{2r_0}.$$

Solving the eigenvalue problem of equation (18) we find that the natural frequencies are

$$\omega_1 = \sqrt{\frac{3P_\infty\gamma}{\rho}(A - B)} \tag{26c}$$

$$\omega_2 = \sqrt{\frac{3P_\infty\gamma}{\rho}(A + B)} \tag{26d}$$

The two eigenvectors can be given as the columns of a square matrix $\mathbf{A}_2$:

$$\mathbf{A}_2 = \begin{pmatrix} 1 & 1 \\ 1 & -1 \end{pmatrix} \tag{26e}$$

The first mode describes two bubbles oscillating in phase; in the second mode the bubbles are 180° out of phase. Both the phasing and the relative amplitudes of the bubble oscillation in these two normal modes could, of course, have been anticipated from the problem's symmetry.

Exact analytic solutions of $\mathbf{M}$ for more complex configurations are not available. However, a very accurate approximate expression for $\mathbf{M}$ may be derived for spherical bubbles by assuming a uniform source distribution dq' on each bubble's surface. In any multiple bubble configuration this assumption implies a nonuniform potential field ϕ' on each bubble. It follows from equation (7) that

$$T_1 < -\frac{1}{2}\rho \int_A \phi' \frac{\partial\phi'}{\partial n}\, dA. \tag{27}$$

The definition of the divergence, implies

$$dq' = \nabla^2\phi' dV. \tag{28}$$

Equations (27) and (28) and Green's theorem then yield

$$T_1 < -\tfrac{1}{2}\rho \int \phi' dq'. \tag{29}$$

Integrating

$$T_1 < \tfrac{1}{2}\delta\mathbf{v}^T\mathbf{M}\delta\mathbf{v} \tag{30}$$

where the approximate mass matrix is

$$M_{ij} = \frac{\rho}{4\pi r_{ij}}. \tag{31}$$

Here r_{ij} is the center-to-center separation between the ith and jth bubble if $i \neq j$, and the equilibrium bubble radius if $i = j$.

Use of the approximate mass matrix leads to an overestimate of T_1, and thus underestimates ω (equation (19)). The accuracy of the approximation is expected to increase with increasing bubble separation. The examples in the following section demonstrate that equation (31) provides a remarkably accurate estimate of ω even for bubbles in close proximity.

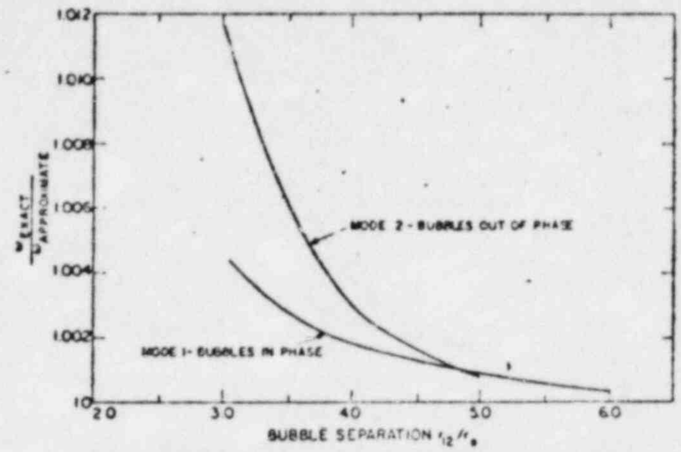

Fig. 2 Exact and approximate frequencies for two bubbles

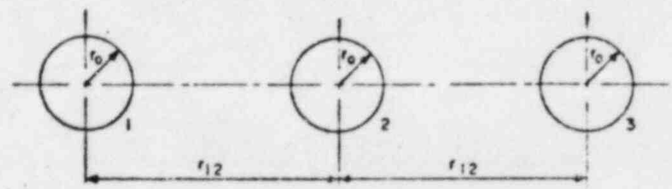

Fig. 3 Three-bubble configuration

Multiple Bubble Configurations

Consider again two spherical bubbles of equal radii. Using equation (31) for $\mathbf{M}$ and taking $\lambda = 1/\omega^2$, equation (18) reduces to the eigenproblem

$$\begin{vmatrix} \dfrac{\lambda P_\infty\gamma}{V_0} - \dfrac{\rho}{4\pi r_0} & \dfrac{-\rho}{4\pi r_{12}} \\ \dfrac{-\rho}{4\pi r_{12}} & \dfrac{\lambda P_\infty\gamma}{V_0} - \dfrac{\rho}{4\pi r_0} \end{vmatrix} = 0. \tag{32}$$

This yields natural frequencies

$$\omega_1 = \frac{1}{r_0}\sqrt{\frac{3p_\infty\gamma}{\rho\left(1 + \dfrac{r_0}{r_{12}}\right)}} \tag{33a}$$

and

$$\omega_2 = \frac{1}{r_0}\sqrt{\frac{3P_\infty\gamma}{\rho\left(1 - \dfrac{r_0}{r_{12}}\right)}} \tag{33b}$$

with the mode shapes defined by the columns of $\mathbf{A}_2$

$$\mathbf{A}_2 = \begin{pmatrix} 1 & 1 \\ 1 & -1 \end{pmatrix} \tag{33c}$$

Fig. 2 compares these approximate solutions (equations (33a) and (33b)) with the exact solutions (equation (26)). The approximate solutions underestimate ω for both mode shapes, as expected. How-

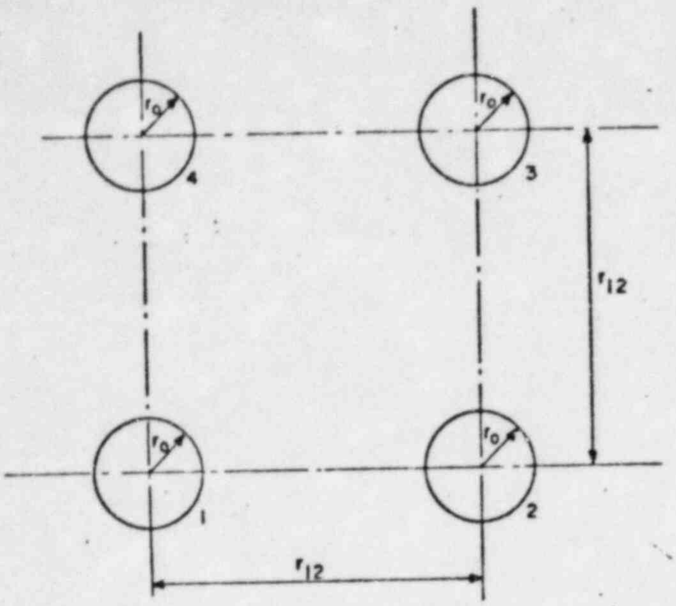

Fig. 4 Four-bubble configuration

ever, for bubble center-to-center separations as low as three bubble radii, the error is less than 1.2 percent.

Consider three aligned bubbles with center-to-center separations r_{12} (Fig. 3). The approximate **M** matrix is

$$\mathbf{M} = \frac{\rho}{4\pi}\begin{pmatrix} \frac{1}{r_0} & \frac{1}{r_{12}} & \frac{1}{2r_{12}} \\ \frac{1}{r_{12}} & \frac{1}{r_0} & \frac{1}{r_{12}} \\ \frac{1}{2r_{12}} & \frac{1}{r_{12}} & \frac{1}{r_0} \end{pmatrix} \tag{34}$$

The eigenvalues are found to be

$$\omega_1 = \frac{1}{r_0}\sqrt{\frac{3P_\infty\gamma}{\rho\left(1 + 1.6861\frac{r_0}{r_{12}}\right)}} \tag{35a}$$

$$\omega_2 = \frac{1}{r_0}\sqrt{\frac{3P_\infty\gamma}{\left(1 - 0.5\frac{r_0}{r_{12}}\right)}} \tag{35b}$$

$$\omega_3 = \frac{1}{r_0}\sqrt{\frac{3P_\infty\gamma}{\left(1 - 1.1861\frac{r_0}{r_{12}}\right)}} \tag{35c}$$

and the mode shapes are given by the columns of $\mathbf{A}_3$

$$\mathbf{A}_3 = \begin{pmatrix} 1 & 1 & 1 \\ 1.1861 & 0 & -1.6861 \\ 1 & -1 & 1 \end{pmatrix}. \tag{35d}$$

In the first mode, all the bubbles oscillate in phase. In the second, only the two outside bubbles have finite amplitudes of oscillation. The third mode has the center bubble 180° out of phase with the other two. As expected from the symmetry of the configuration, the end bubbles have the same amplitude in each mode. Comparison with the results of a numerical calculation (in which the mass matrix was accurately computed by a method of images analysis) shows that the estimated natural frequencies are in error by less than 1.7 percent for $r_{12}/r_0 \geq 3$ (11).

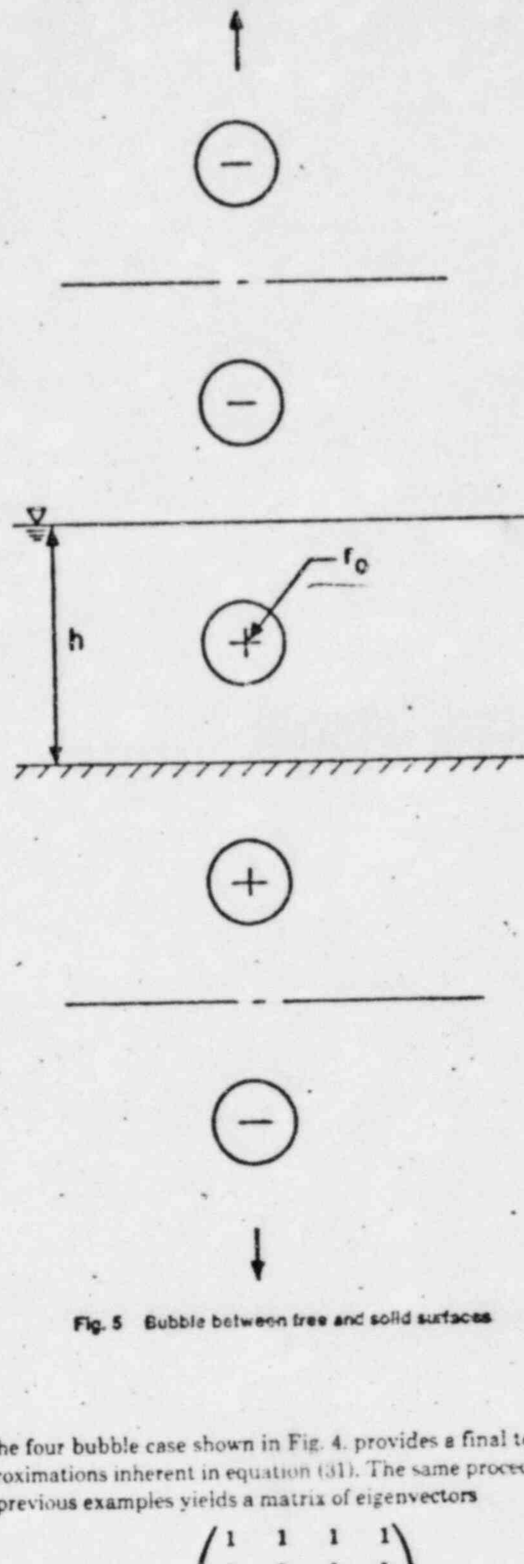

Fig. 5 Bubble between free and solid surfaces

The four bubble case shown in Fig. 4. provides a final test of the approximations inherent in equation (31). The same procedure as in the previous examples yields a matrix of eigenvectors

$$\mathbf{A}_4 = \begin{pmatrix} 1 & 1 & 1 & 1 \\ 1 & 1 & -1 & -1 \\ 1 & -1 & 1 & -1 \\ 1 & -1 & -1 & 1 \end{pmatrix} \tag{36a}$$

Again, the symmetry of the problem requires that each bubble have the same amplitude and that the normal modes involve phase differences of 0° or 180°. The eigenvectors actually found are consistent with both of these requirements.

The eigenvalues are found to be

$$\omega_1 = \frac{1}{r_0}\sqrt{\frac{3P_\infty\gamma}{\rho\left(1 + 2.707\frac{r_0}{r_{12}}\right)}} \tag{36b}$$

$$\omega_2 = \omega_4 = \frac{1}{r_0}\sqrt{\frac{3P_\infty\gamma}{\rho\left(1 - 0.707\,\frac{r_0}{r_{12}}\right)}} \qquad (36c)$$

$$\omega_3 = \frac{1}{r_0}\sqrt{\frac{3P_\infty\gamma}{\rho\left(1 - 1.293\,\frac{r_0}{r_{12}}\right)}}. \qquad (36d)$$

The identity of ω_2 and ω_4 results from the geometric identity of mode shapes 2 and 4 (the second and fourth columns of $\mathbf{A}_1$). These results are within 2.7 percent of an accurate numerical calculation for $r_{12}/r_0 = 3$ (11).

Free and Solid Surfaces

The normal modes for $2N$ bubbles in an infinite pool will implicitly determine the normal modes for N bubbles adjacent to a plane solid or free surface that replaces any plane of symmetry in the $2N$ bubble configuration. A first example is presented in equations (33a) and (33b) which can be interpreted as the frequencies of oscillation of a single bubble adjacent to a solid or free surface, respectively. Two bubbles aligned parallel to a free surface, twice as far apart as their distance from the surface, will have two normal modes corresponding to Modes 2 and 3 of the four bubble configuration, equations (36c) and (36d). If the surface is replaced by a solid surface, the two normal modes are given by Modes 1 and 4. These results demonstrate expected trends: proximity to a free surface raises normal mode frequencies and brings them closer together; a solid surface lowers frequencies and spreads them apart.

The analysis of bubbles oscillating near free and solid surfaces is greatly simplified by a method of images approach. When more than one free or solid surface is present, the number of image bubbles required is usually infinite, but the amplitudes and phasing of the image bubbles can be deduced in advance by inspection. Images are always of the same amplitude as the "object" bubble, in phase for a solid surface image and 180° out of phase for a free surface. When the mode shape can be determined directly, an analysis of the natural frequencies based on equation (19) is considerably simpler than an attempt to identify the appropriate eigenvalues in equation (18) when the dimensions of $\mathbf{M}$ and $\mathbf{K}$ are infinite.

For a system with N independent bubbles and an arbitrary boundary of free and solid plane surfaces there are N normal modes of oscillation. The problem can always be reduced to analyzing an $N \times N$ mass matrix whose M_{ij} element relates the potential on the ith bubble to the volume flux from the jth bubble *and* all its images. If the $j + kN$th bubble is a reflection of the jth,

$$M_{ij} = \sum_{k=0}^{\infty}\left[\frac{\rho}{4\pi r_{i,j+kN}}\left(\frac{q_{j+kN}}{q_j}\right)\right]. \qquad (37)$$

The ratio q_{j+kN}/q_j will equal ± 1 depending on the types of "reflecting" surfaces involved. The elements of the stiffness matrix $\mathbf{K}$ will be given by equation (15) as before.

Fig. 5 shows an example of a single bubble situated midway between a free and solid surface separated by a distance h. For this case,

$$M = \frac{\rho}{4\pi r_0}\left[1 + \sum_{k=1}^{\infty}(-1)^k\frac{r_0}{kh}\right] \qquad (38)$$

and equation (19) then yields

$$\omega = \frac{1}{r_0}\sqrt{\frac{3P_\infty\gamma}{\rho\left(1 + \sum_{k=1}^{\infty}(-1)^k\left(\frac{r_0}{kh}\right)\right)}} = \frac{1}{r_0}\sqrt{\frac{3P_\infty\gamma}{\rho\left(1 - 0.693\,\frac{r_0}{h}\right)}}. \qquad (39)$$

Cole [6] and Friedman [7] use a similar technique to describe the motion of an explosively formed bubble in a pool of finite depth.

Concluding Remarks

The small amplitude oscillations of a single spherical bubble in an unbounded liquid are analogous to those of a one-spring, one-mass system with $K = 3P_\infty\gamma/V_0$ and $M = \rho/4\pi r_0$. Nonspherical bubble shape has only a weak effect on predicted frequency. Multiple spherical bubbles in an unbounded liquid have normal mode frequencies and mode shapes that can be found by formulating K and M matrices as outlined in this paper and analyzing the matrix $M^{-1}K$. Use of approximate mass matrix elements greatly simplifies the construction of M with little loss of accuracy in the frequency predictions. The effects of plane pool boundaries on the elements of M can be determined by a method of images.

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Survey of Thermal, Radiation, and Viscous Damping of Pulsating Air Bubbles in Water*

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A theoretical discussion is presented on the fundamental processes by which pulsating gas bubbles in liquids dissipate their energy. The survey is limited to the case where the amplitude of the volume pulsations are assumed to be sufficiently small that the pulsations may be described by linear equations. A portion of the energy of the bubble system is lost by the radiation of spherical sound waves, a part is lost by heat conduction due to the polytropic compressions and expansions of the enclosed gas, and a portion is lost by viscous dissipation attributed to viscous forces acting at the gas-liquid interface. A survey is made of the procedures for measuring the resonant damping constant as described in the methods of successive oscillations, width of the resonance response, standing-wave ratios, and resonance absorption. Experimental results verify that the damping at resonance is due to thermal and radiation, and possibly viscous damping.

GLOSSARY OF SYMBOLS

A attenuation
a_1 $R_0 P'/\rho_1 s_{r1}$
B $b\dot{r}^2/2$ Rayleigh's dissipation function
b dissipation coefficient
C generalized driving function
c velocity of sound
D thermal diffusivity
d thickness of bubble screen
E_i incident sound energy
E_r reflected sound energy
F characteristic frequency of the gas bubble
f_M Minnaert's resonant frequency
f_0 resonant frequency
G universal gas constant
g a factor which takes into account the effect of surface tension
h height above the bubble producers
I idem factor
K thermal conductivity
k restoring stiffness
L Lagrangian function
m_1 mass of the gas in the bubble
m_2 generalized mass
m' mass of the gas contained in the volume v'
n_0 average number of bubbles per unit volume
P_0 static pressure
P_2 instantaneous pressure in the undisturbed liquid
P' complex amplitude of sinusoidal pressure p'
P_2' instantaneous pressure on the bubble surface
P_i pressure inside the bubble
p sinusoidal pressure on the liquid surface
p' sinusoidal pressure on the bubble surface
p_a acoustic pressure on the bubble surface
p_i incident sound pressure
p_r reflected sound pressure
Q number of cycles required for the amplitude of motion to reduce to $e^{-\pi}$ of its original value
q amount of heat energy transferred
R radial distance
R_0 mean bubble radius
R_1 instantaneous bubble radius
R' nonresonant bubble radius
r change in radius from the mean bubble radius
S net stress dyadic
s_p specific heat at constant pressure
s_v specific heat at constant volume
T_0 equilibrium absolute temperature
T absolute temperature
t time
U internal energy
V_0 equilibrium bubble volume
V_1 instantaneous bubble volume
W work done on the bubble
X rate of pure strain dyadic
y amplitude of the change in temperature from the equilibrium temperature
Z tube radius
α a factor which describes the departure of the bubble stiffness from the adiabatic stiffness
β $[(R_0/R')^2-1]^2$
γ ratio of specific heats
δ $1/Q$ damping constant
δ_0 resonant damping constant
ϵ angle between the incident sound ray and the normal to the bubble screen
η polytropic exponent
θ change in temperature from the equilibrium temperature
Λ natural logarithmic decrement
λ wavelength
μ coefficient of viscosity
ρ density
σ surface tension
v change in volume from the equilibrium bubble volume
v' infinitesimal element of volume in the gas bubble
ϕ $(\omega/2D)^{\frac{1}{2}}$
ψ $(j\omega/D)^{\frac{1}{2}}$
Ω velocity potential
ω circular frequency

* Based on a thesis submitted to the Faculty of the Columbian College of The George Washington University in partial satisfaction of the requirements for the degree of Master of Science.

I. INTRODUCTION

THE earliest reference to bubbles as sound sources was made by Bragg,[1] who attributed to entrained ...ebbles the murmuring of a brook and the "plunk" ...plets falling into water. Minnaert[2] has since ...n that the sound generated by gas bubbles in ...ds is associated with simple volume pulsations of ... bubble without change of shape. The bubble ...aves as a simple damped oscillating system with ... degree of freedom. Therefore, the differential ...tion of motion for the bubble system has the same ... as the second-order linear differential equation ... mass fastened to a spring. As the bubble periodi... expands and contracts, the surrounding liquid is ... ...ert mass which is set into vibration, while the ...ess is due to the gas in the bubble. This zero-...er radiator has a sharply defined resonance at the ...uency

$$f_M = (3\gamma P_0/\rho_2)^{\frac{1}{2}}/2\pi R_0, \qquad (1)$$

...re R_0 is the mean radius of the bubble, P_0 is the ... pressure at which the bubble has the mean radius, ... is the ratio of the specific heats of the gas enclosed ...e bubble, and ρ_2 is the density of the liquid. This ...nant frequency derived by Minnaert assumes an ...batic equation of state for the gas in the bubble.

...e volume pulsation frequency of nonspherical gas ...bles in liquids has been considered by Strasberg[3] ... used oblate spheroids to approximate the non-...rical shapes. This determination indicates that the ...uency is only slightly dependent upon the ratio of ... major to the minor axis of the spheroid. In fact, ... a ratio of two, the volume pulsation frequency of ... oblate spheroid differs by only 2% from that of a ...ere with the same volume. Observations have shown ... large bubbles are generally nonspherical while ... small bubbles tend to be spherical.

...n addition to simple volume pulsations, there also ... be oscillations in the shape of the bubble. The ...ural frequency for the higher modes of shape ...ation has been calculated by Lamb[4]; Strasberg[5] ... used this analysis to demonstrate that shape ...ations do not seem to result in significant sound ...ure except perhaps very close to the bubble.

...e sound pressure resulting from excitation of ...me pulsations by several mechanisms has recently ... discussed in the literature.[5] The mechanisms, ...ch cause bubbles to pulsate and radiate sound, are ...le formation, coalescence, or division; the motion ... free stream of liquid containing entrained gas ...bles past an obstacle, or the flow of liquid containing entrained bubbles through a pipe past a constriction; and an incident sound wave.

Experiments conducted by Sörensen[6] showed that liquids containing a gas possess higher sound damping characteristics than do those which are gas free. Just a few widely dispersed bubbles, which are so small as to be invisible, can have an appreciable acoustic effect. When a large number of these small bubbles are present, the liquid will be nearly opaque acoustically. Small impurities in liquids, such as suspended particles, have negligible influence in comparison with the damping increase due to bubbles. Therefore, bubbles have a considerable importance in the transmission of underwater sound. In order to understand the attenuation of sound by gas bubbles in liquids, the fundamental processes by which pulsating bubbles dissipate their energy must be known. This discussion will investigate the portion of the energy radiated in the form of spherical sound waves, the part which is transformed into heat energy during the polytropic compressions and expansions of the enclosed gas, and the part of the energy lost in viscous dissipation. It may be that these three processes completely account for the total damping of gas bubbles in liquids.

II. THEORY

(a) Equation of Motion

Periodic enforced changes in the pressure on a bubble result in volume pulsations of the bubble. If the amplitude of the volume pulsation is small, the motion of the bubble system is described by a second-order linear differential equation. For this system possessing one degree of freedom, the condition of the bubble system is defined by the change in volume v from the equilibrium volume V_0. The instantaneous volume V_1 of the bubble is the algebraic sum of the mean volume V_0 and v. In a similar manner, the instantaneous radius R_1 of the bubble is the algebraic sum of the mean radius R_0 and the radial increment r. The bubble is assumed to be in an incompressible liquid. At the surface of the liquid, a sinusoidal pressure p is applied:

$$p = P\exp(j\omega t), \qquad (2)$$

where P is a constant. Liquids are slightly compressible, but, as long as the bubble size is small compared to the wavelength of the pressure wave, the liquid is considered incompressible.[7] The instantaneous pressure P_2 in the undisturbed liquid is the sum of the sinusoidal pressure $P\exp(j\omega t)$ and the static pressure P_0:

$$P_2 = P\exp(j\omega t) + P_0. \qquad (3)$$

However, at the bubble surface, the instantaneous pressure P_2' is the instantaneous pressure P_2 in the undisturbed liquid minus the inertial reaction of the

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1 Bragg, *The World of Sound* (G. Bell and Sons Ltd., London,

2 Minnaert, Phil. Mag. 26, 235 (1933).

3 Strasberg, J. Acoust. Soc. Am. 25, 536 (1953).

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5 Strasberg, J. Acoust. Soc. Am. 28, 20 (1956).

6 C. Sörensen, Ann. Physik 26, 121 (1936).

7 See reference 4, Sec. 290.

liquid in motion about the bubble. For the moment, until the inertial reaction of the liquid is determined, the instantaneous pressure P_2' *at the bubble surface* is, defined as the sum of the sinusoidal pressure p' and the static pressure P_0:

$$P_2' = p' + P_0 = P' \exp(j\omega t) + P_0, \tag{4}$$

where P' is the complex amplitude of the driving pressure p'. The bubble, which is in this uniform but alternating pressure field, cannot be in equilibrium with this oscillating pressure unless the bubble itself is pulsating. Uniform pressure in the gas bubble implies that the inertia of the gas is negligible. The liquid surrounding the bubble provides the inertia for the bubble system. The equation of motion for the bubble system is derived in terms of generalized coordinates by using Lagrange's equation. When there are no dissipation or forcing pressures present, Lagrange's equation is

$$(d/dt)(\partial L/\partial \dot{v}) - \partial L/\partial v = 0, \tag{5}$$

where the Lagrangian function L is defined as the kinetic energy minus the potential energy of the system. When dissipation is present, the dissipation pressure is assumed to be proportional to the bubble volume velocity $\dot{v}$. Dissipation of this type may be derived in terms of a function B, known as Rayleigh's dissipation function, and defined as[8]

$$B = [b(\dot{v})^2]/2, \tag{6}$$

where b is the dissipation coefficient. The equation of motion for the bubble system when there is dissipation and a generalized driving function C, where neither arise from a potential, is

$$(d/dt)(\partial L/\partial \dot{v}) - \partial L/\partial v + \partial B/\partial \dot{v} = C. \tag{7}$$

The potential energy of the bubble system is obtained by assuming the gas in the bubble undergoes an adiabatic process during the volume pulsations of the bubble:

$$P_2' V_1^{\gamma} = P_0 V_0^{\gamma} \tag{8}$$

$$\partial P_2' = -(\gamma P_0/V_0)\partial V_1 \tag{9}$$

$$P_2' - P_0 = -(\gamma P_0/V_0)v. \tag{10}$$

Therefore, the potential energy is

$$\text{P.E.} = -\int_0^v (P_2' - P_0)dv = (\gamma P_0/2V_0)v^2. \tag{11}$$

As the bubble periodically expands and contracts, the surrounding liquid is set into vibration. The maximum kinetic energy of the liquid particles occurs at the moment the bubble has again recovered its equilibrium volume V_0. The flow of the liquid is irrotational; therefore, a velocity potential exists. The velocity potential of a liquid particle at a distance R, due to a simple source, in a liquid at rest at infinity, is[9]

$$\Omega = (\dot{v}/4\pi R) \tag{12}$$

and the velocity of this liquid particle is

$$\dot{R} = -\nabla\Omega = (\dot{v}/4\pi R^2). \tag{13}$$

The kinetic energy of all the liquid volume elements of density ρ_2 is

$$\text{K.E} = (\rho_2/2)\int_{R_0}^{\infty} (\dot{R})^2(4\pi R^2)dR. \tag{14}$$

The integration is extended to infinity because the bubble is assumed to be surrounded by a very large liquid volume. Upon integration, the foregoing expression yields the kinetic energy as

$$\text{K.E.} = (\rho_2/8\pi R_0)(\dot{v})^2. \tag{15}$$

Accordingly, the Langrangian L is

$$L = (\rho_2/8\pi R_0)(\dot{v})^2 - (\gamma P_0/2V_0)v^2, \tag{16}$$

and the equation of motion for the bubble system when a sinusoidal pressure $P \exp(j\omega t)$ is applied at the surface of the liquid and dissipation is present, is

$$(\rho_2/4\pi R_0)\ddot{v} + b\dot{v} + (3\gamma P_0/4\pi R_0^3)v = -P\exp(j\omega t). \tag{17}$$

The forcing function $P\exp(j\omega t)$ is preceded by a minus sign as a decrease in pressure results in an increase in the bubble volume. The term $(\rho_2/4\pi R_0)$ is the generalized mass m_2 of the bubble system. The stiffness of the bubble system is defined as the change in pressure on the bubble surface associated with the change in bubble volume; therefore, the term $(\gamma P_0/V_0)$ is the adiabatic stiffness k_{ad},

$$k_{ad} = -(\partial P_2'/\partial V_1) = (\gamma P_0/V_0). \tag{18}$$

Consequently, the linear second-order differential equation of motion for the bubble system is also written as

$$m_2\ddot{v} + b\dot{v} + k_{ad}v = -P\exp(j\omega t). \tag{19}$$

When the bubble is slightly nonspherical, each term in Eq. (19) is nearly independent of shape when the mean radius R_0 is taken as the radius of a sphere of the same volume.[2] Transient volume pulsations are given by the solution of Eq. (19) when the right side of the equation is set equal to zero. Furthermore, if the dissipation is negligible, Eq. (19) becomes

$$m_2\ddot{v} + k_{ad}v = 0 \tag{20}$$

and the resonant frequency of the bubble system is

$$f_M = (1/2\pi)(k_{ad}/m_2)^{\frac{1}{2}} = (1/2\pi R_0)(3\gamma P_0/\rho_2)^{\frac{1}{2}} \tag{21}$$

which is Minnaert's expression as given in Eq. (1).

[8] H. Goldstein, *Classical Mechanics* (Addison-Wesley Publishing Company, Inc., Cambridge, 1953), p. 21.

[9] See reference 4, Sec. 56.

When the right side of Eq. (19) is zero, i.e., the driving pressure has been removed, the sound pulse from the bubble consists of a damped exponential sinusoidal oscillation. The number of cycles required for the amplitude of motion to reduce to $e^{-\pi}$ of its original value is the Q of the bubble system. When the dissipation is small, the difference between the frequency of the oscillation and the resonant frequency of the bubble system without dissipation is negligible; the Q of the bubble system is expressed as

$$Q=2\pi f_0 m_2/b, \tag{22}$$

where f_0 is the resonant frequency. The fraction of the remaining energy lost in each cycle of the bubble oscillation is just $2\pi/Q$. In order to maintain a constant amplitude of oscillation with time for forced oscillations of a bubble, the rate at which energy is supplied to the bubble system must be just equal to the rate at which the energy of this system is dissipated. For forced oscillations of a bubble, the Q may be alternately defined as

$$Q=f_0/(f_2-f_1), \tag{23}$$

where f_2 and f_1 are the two frequencies respectively above and below resonance at which the average sound power of the bubble has dropped to one-half its resonance value.[10] The total damping constant δ is now defined as the reciprocal Q of the bubble system or π^{-1} times the natural logarithmic decrement Δ:

$$\delta=1/Q=\Delta/\pi. \tag{24}$$

The total damping may be explained by losses originating from three processes:

1. Thermal damping δ_{th} due to the thermal conduction between the gas in the bubble and the surrounding liquid.
2. Sound radiation damping δ_{rad}.
3. Viscous damping δ_{vis} due to viscous forces at the gas-liquid interface.

The total damping constant δ is the sum of these three processes:

$$1/Q=\delta=\delta_{th}+\delta_{rad}+\delta_{vis}. \tag{25}$$

(b) Thermal Damping

In the derivation of Eq. (1) for the resonant frequency, the adiabatic equation of state was assumed. The pressure and volume changes are in phase with one another so that dP'_2/P_0 equals $-\gamma dV_1/V_0$. For the adiabatic case, there is no transfer of heat. In the other limiting case of a purely isothermal process in the gas space, the pressure and volume changes are again in phase; dP'_2/P_0 equals $-dV_1/V_0$. For this case, there is just as much heat flowing outward from the bubble during compression as flows inward during expansion. The work done by the driving pressure in compressing the gas space is just equal to the work done by the expanding gas in moving the surrounding liquid. However, for the case of a real bubble, the gas in contact with the liquid closely follows the isothermal equation of state since the liquid has a large specific heat and thermal conductivity. In the center of a real bubble away from a substance having a high specific heat, the gas nearly follows an adiabatic equation of state. Therefore, the thermal process is polytropic for a real bubble, and a phase difference exists between the increase in pressure per unit original pressure and the decrease in volume per unit original volume. This phase difference causes a hysteresis effect. The work done on the gas volume by the driving pressure during compression is more than the work done by the gas space in moving the surrounding liquid during expansion. This difference in the work done represents a net flow of heat into the liquid. This net flow of heat into the liquid is characterized by the thermal damping constant.

The subject of thermal damping has been investigated independently by Pfriem,[11] Willis,[12] and Saneyosi[13] and all have obtained somewhat similar results. The results of both Willis and Saneyosi are available, but unfortunately their derivations are not easily accessible. Consequently, the derivation as outlined by Pfriem will be more or less followed.

In deriving the expression for the thermal damping constant, the gas bubble is assumed to be in an incompressible liquid, and is excited to volume pulsations by a sinusoidal pressure $P'\exp(j\omega t)$ applied at the surface of the bubble. The liquid has a large specific heat and thermal conductivity, and behaves as a heat reservoir. Consequently, in the liquid adjacent to the gas-liquid interface, it will be assumed that there are no changes in temperature. In addition, the temperature in the center of the bubble is finite. The oscillations in the pressure, volume, and temperature of the gas in the bubble will be assumed small. Consequently, the equations relating these three thermodynamic coordinates are linear. In addition, the density and the specific heats of the gas are regarded as constant. In the gas, the pressure is not a function of position but only of time. Therefore, the gas is in a uniform but alternating pressure field; the inertia of the gas is negligible. The heat transfer process is conduction. Convection is unimportant as the time factor for establishment of this process is considerably larger than the time consumed during a half-cycle compression of the bubble.

In order to calculate the thermal contributions to the total damping of the bubble system, the change in

[10] L. Kinsler and A. Frey, *Fundamentals of Acoustics* (John Wiley & Sons, Inc., New York, 1950), p. 24.

[11] H. Pfriem, Akust. Z. 5, 202 (1940).

[12] L. Spitzer, "Acoustical properties of gas bubbles in a liquid," Columbia University Office of Scientific Research and Development, Rept. 1705, Sec. No. 6.1 sr 20 918, July 15, 1943.

[13] Z. Saneyosi, Electrotech. J. 5, 49 (1941).

bubble volume must first be determined. When the driving pressure at the bubble surface compresses the bubble, work is done on the gas space. This work done on the gas space increases the internal energy of the gas, and also results in a transfer of heat energy through the gas. The added heat is transferred by conduction from the gas bubble into the surrounding liquid. The compression process must obey the conservation of energy principle as stated in the first law of thermodynamics:

$$\Delta U = \Delta q + \Delta W, \tag{26}$$

where ΔU is the increase in internal energy of the gas space, Δq is the heat added to the gas space, and ΔW is the work done on the gas space. When each term in Eq. (26) is divided by an infinitesimal time Δt and Δt is allowed to approach zero as a limit, the rate of increase in internal energy is given as

$$dU/dt = dq/dt + dW/dt. \tag{27}$$

At a point in the gas space, the rate at which work is done per unit volume by the driving pressure on an infinitesimal element of volume v' of the gas is

$$dW/dt = -(P_2'/v')(\partial v'/\partial t). \tag{28}$$

For this small element of volume at a point, the rate of increase in internal energy per unit volume is

$$dU/dt = \rho_1 s_{v1}(d\theta_1/dt), \tag{29}$$

where s_{v1} is the specific heat of the gas at constant volume, and θ_1 is the change in gas temperature from the equilibrium absolute temperature T_0. The rate of transfer of heat energy per unit volume for an infinitesimal volume at a point as a result of conduction is proportional to the divergence of the temperature gradient; the proportionality constant is the thermal conductivity K_1 of the gas:

$$\partial q/\partial t = K_1 \nabla^2 \theta_1. \tag{30}$$

Since the temperature is a function of time and radial distance, Eq. (30) becomes

$$\partial q/\partial t = (K_1/R)[\partial^2(R\theta_1)/\partial R^2], \tag{31}$$

where the spherical coordinate system originates at the center of the bubble. When the above expressions are substituted into Eq. (27), the differential equation becomes

$$(\rho_1 s_{v1}/R)[\partial(R\theta_1)/\partial t] = (K_1/R)[\partial^2(R\theta_1)/\partial R^2] - (P_2'/v')(\partial v'/\partial t). \tag{32}$$

A substitution can be made for the second term on the right side of the foregoing equation by considering the ideal gas equation:

$$P_2'v' = m'G(T_0 + \theta_1), \tag{33}$$

where m' is the mass of the gas contained in the volume element v', and G is the universal gas constant. Equation (33) may be differentiated with respect to time to yield

$$P_2'(\partial v'/\partial t) = (m'G/R)[\partial(R\theta_1)/\partial t] - v'(\partial P_2'/\partial t) \tag{34}$$

$$(P_2'/v')(\partial v'/\partial t) = (\rho_1 G/R)[\partial(R\theta_1)/\partial t] - \partial P_2'/\partial t \tag{35}$$

$$(P_2'/v')(\partial v'/\partial t) = (\rho_1 s_{p1}/R)[\partial(R\theta_1)/\partial t] - (\rho_1 s_{v1}/R)[\partial(R\theta_1)/\partial t] - j\omega P' \exp(j\omega t), \tag{36}$$

where s_{p1} is the specific heat at constant pressure. Therefore, the linear differential equation describing the temperature field within the gas bubble is

$$\partial(R\theta_1)/\partial t = D_1[\partial^2(R\theta_1)/\partial R^2] + j(\omega R/\rho_1 s_{p1})P' \exp(j\omega t), \tag{37}$$

where the thermal diffusivity D_1 is $K_1/\rho_1 s_{p1}$. A solution of this differential equation must satisfy the boundary conditions. At the center of the bubble, the change in temperature θ_1 must be finite, and the gradient of the change in temperature must be zero. The change in temperature must be zero at the gas-liquid interface while the gradient of the change in temperature must be finite. The solution of Eq. (37) may be obtained by several methods. One method is to assume that the change in temperature θ_1 is

$$\theta_1 = y \exp(j\omega t) \tag{38}$$

where y is a function of R only.† Therefore, Eq. (37) becomes

$$j\omega(Ry) = D_1[\partial^2(Ry)/\partial R^2] + j\omega RP'/\rho_1 s_{p1}. \tag{39}$$

The solution of Eq. (39) is

$$Ry = a_1[R/R_0 - \sinh(\psi_1 R)/\sinh(\psi_1 R_0)], \tag{40}$$

where a_1 is $R_0P'/\rho_1 s_{p1}$ and ψ_1 is $(j\omega/D_1)^{\frac{1}{2}}$. Therefore the temperature field within the whole gas space is now known. The change in bubble volume v can now be determined for a change in the driving pressure at the bubble surface and a change in the temperature. The total change in volume of the gas space is the sum of the changes in volume of all concentric shells whose radius is R and thickness dR. A shell of thickness dR

† It is evident that Eq. (38) is not an exact solution of the physical problem, although it satisfies all the conditions of the mathematical equations. The physical reasoning indicates that the mean temperature inside the bubble must increase toward the center of the bubble. However, according to Eq. (38), at the bubble surface, the temperature gradient is $(dy/dR)\exp(j\omega t)$ with a time average value of zero, whereas its time average value should be negative for a net outward flow of heat. This discrepancy comes about in neglecting certain second order terms in order to obtain a differential equation with linear coefficients, and also in assuming that θ_1 and v are sinusoidal. The need for an increase in the mean value of θ_1 indicates also the existence of higher even harmonics. However, the subsequent treatment of the solution of Eq. (38) leads to results that are quite good. It is possible to go back and correct the equations by the method of successive approximations.

as a volume

$$v_0=4\pi R^2dR. \tag{41}$$

In accordance with the ideal gas law, two different states of the gas are expressed as

$$P_2'v=P_0v_0T/T_0, \tag{42}$$

where T is the absolute temperature of the compressed gas. When Eq. (42) is differentiated, the result is

$$dv=(v_0/T_0)dT-(v_0/P_0)dP_2' \tag{43}$$

$$dv=4\pi\exp(j\omega t)[R^2y/T_0-R^2P'/P_0]dR. \tag{44}$$

The expression for Ry is obtained from Eq. (40). Therefore, the total change in the bubble volume is

$$v=4\pi e^{j\omega t}\int_0^{R_0}\times\left\{\frac{R_0P'}{\rho_1s_{p1}T_0}\left[\frac{R^2}{R_0}-R\frac{\sinh(\psi_1R)}{\sinh(\psi_1R_0)}\right]-\frac{P'R^2}{P_0}\right\}dR \tag{45}$$

$$=\frac{V_0P'e^{j\omega t}}{\rho_1s_{p1}T_0}\times\left[1-\frac{\rho_1s_{p1}T_0}{P_0}-\frac{3}{\psi_1^2R_0^2}\left\{[\psi_1R_0\coth(\psi_1R_0)]-1\right\}\right]. \tag{46}$$

Equation (46) is further simplified by noting that $T_0=P_0/\rho_1(s_{p1}-s_{v1})$. Therefore, v is

$$v=-\frac{V_0P'e^{j\omega t}}{\gamma P_0}\times\left[1+\frac{3(\gamma-1)}{\psi_1^2R_0^2}\left([\psi_1R_0\coth(\psi_1R_0)]-1\right)\right] \tag{47}$$

$$-\frac{v}{V_0}=\frac{V_0-V_1}{V_0}=\frac{P_2'-P_0}{\gamma P_0}\times\left[1+\frac{3(\gamma-1)}{\psi_1^2R_0^2}\left([\psi_1R_0\coth(\psi_1R_0)]-1\right)\right]. \tag{48}$$

When the change in volume per unit original volume $(V_0-V_1)/V_0$ is plotted against the change in pressure per unit original pressure $(P_2'-P_0)/P_0$ on a pressure-volume graph for the real components of Eq. (48), the area enclosed by the compression and expansion curves represents the net loss of energy by heat conduction. The change in bubble volume is now known. There remains now the task of relating the change in the bubble volume and the assumed harmonic excitation pressure at the surface of the bubble to the vibrational properties of the gas bubble, i.e., the stiffness and damping attributes.

In order to determine the stiffness and damping, the differential equation of motion for the bubble system is considered. Since the inertia of the gas in the bubble is negligible, Eq. (19) can be rewritten for the differential equation of motion for the total gas space in the bubble

$$b_{th}\dot{v}+kv=-[P\exp(j\omega t)+m_2\ddot{v}]=-P'\exp(j\omega t), \tag{49}$$

where $P'\exp(j\omega t)$ is the sinusoidal excitation pressure at the surface of the bubble. When the expression for the change in bubble volume v, as given by Eq. (47), is differentiated and substituted into Eq. (49), the following expression is obtained:

$$\frac{1}{k+j\omega b_{th}}=\frac{V_0}{\gamma P_0}\times\left[1+\frac{3(\gamma-1)}{\psi_1^2R_0^2}\left([\psi_1R_0\coth(\psi_1R_0)]-1\right)\right]. \tag{50}$$

Since the parameter ψ_1 has the symbol j under the square root, which is undesirable, and there is the need to separate Eq. (50) into real and imaginary components, a substitution for ψ_1 is introduced:

$$\psi_1^2=(1+j)^2\phi_1^2=2j\phi_1^2=j\omega/D_1$$

$$\psi_1=(1+j)\phi_1=(1+j)(\omega/2D_1)^{\frac{1}{2}}.$$

Accordingly, the parameter ϕ_1R_0 is $R_0(\omega/2D_1)^{\frac{1}{2}}$ or $R_0(\pi\rho_1s_{p1}f/K_1)^{\frac{1}{2}}$. When the frequency and the radius of the bubble are kept constant, the quantity ϕ_1R_0 varies as the square root of the density of the gas, or alternatively as the square root of the average pressure inside the bubble since the specific heat at constant pressure and the thermal conductivity are independent of density. Another condition exists for ϕ_1R_0 when the excitation frequency is constant and the radius of the bubble satisfies Eq. (1) for a resonant bubble. For this case, the parameter ϕ_1R_0 varies as the average pressure inside the bubble. By introducing the substitution for ψ_1 and noting the identities for $\sinh(\phi_1R_0+j\phi_1R_0)$ and $\cosh(\phi_1R_0+j\phi_1R_0)$, Eq. (50) becomes

$$\frac{k-j\omega b_{th}}{k^2+(\omega b_{th})^2}=\frac{V_0}{\gamma P_0}\left\{1+\frac{3(\gamma-1)}{2\phi_1R_0}\left[\frac{\sinh(2\phi_1R_0)-\sin(2\phi_1R_0)}{\cosh(2\phi_1R_0)-\cos(2\phi_1R_0)}-j\left(\frac{\sinh(2\phi_1R_0)+\sin(2\phi_1R_0)}{\cosh(2\phi_1R_0)-\cos(2\phi_1R_0)}-\frac{1}{\phi_1R_0}\right)\right]\right\}. \tag{51}$$

Even though Eq. (51) is separated into real and imaginary terms, the form is still not suitable for determining the thermal damping constant at resonance. In Eq. (22), the damping constant is given in terms of the dissipation coefficient b, resonant circular frequency ω_0, and the generalized mass m_2. However, the generalized mass is simply the stiffness k divided by the square of the resonant circular frequency. Therefore,

the dimensionless damping constant, which at resonance is the reciprocal Q, is $\omega b/k$. At resonance, the maximum value of the thermal damping constant is about one-tenth; therefore, $(\omega b_{th}/k)^2$ is very small and can be neglected with respect to unity:

$$\frac{(1/k)[1-j(\omega b_{th}/k)]}{[1+(\omega b_{th}/k)^2]}\simeq(1/k)[1-j(\omega b_{th}/k)]. \quad (52)$$

Accordingly, Eq. (51) becomes

$$\frac{1}{k}\left[1-j\left(\frac{\omega b_{th}}{k}\right)\right] \quad (53)$$

$$\simeq\frac{V_0}{\gamma P_0}\left[1+\frac{3(\gamma-1)}{2\phi_1R_0}\left(\frac{\sinh(2\phi_1R_0)-\sin(2\phi_1R_0)}{\cosh(2\phi_1R_0)-\cos(2\phi_1R_0)}\right)\right]$$

$$\times\left[1-j\frac{\left(\dfrac{\sinh(2\phi_1R_0)+\sin(2\phi_1R_0)}{\cosh(2\phi_1R_0)-\cos(2\phi_1R_0)}-\dfrac{1}{\phi_1R_0}\right)}{\left(\dfrac{2\phi_1R_0}{3(\gamma-1)}+\dfrac{\sinh(2\phi_1R_0)-\sin(2\phi_1R_0)}{\cosh(2\phi_1R_0)-\cos(2\phi_1R_0)}\right)}\right].$$

Therefore, the dimensionless thermal damping constant is

$$\frac{\omega b_{th}}{k}\simeq\frac{\dfrac{\sinh(2\phi_1R_0)+\sin(2\phi_1R_0)}{\cosh(2\phi_1R_0)-\cos(2\phi_1R_0)}-\dfrac{1}{\phi_1R_0}}{\dfrac{2\phi_1R_0}{3(\gamma-1)}+\dfrac{\sinh(2\phi_1R_0)-\sin(2\phi_1R_0)}{\cosh(2\phi_1R_0)-\cos(2\phi_1R_0)}}. \quad (54)$$

For large values of $2\phi_1R_0$, i.e., equal to or greater than 5, the thermal damping constant is given to within one percent by

$$\frac{\omega b_{th}}{k}\simeq\frac{1-(1/\phi_1R_0)}{1+[2\phi_1R_0/3(\gamma-1)]}\simeq\frac{3(\gamma-1)}{2\phi_1R_0}. \quad (55)$$

Large values of $2\phi_1R_0$ correspond to large bubbles or the adiabatic case. The dissipation arising from heat conduction vanishes. The thermal damping constant for very small values of $2\phi_1R_0$, i.e., equal to or less than 2, is given to within one percent by

$$\omega b_{th}/k\simeq[(\gamma-1)/\gamma][(2\phi_1R_0)^2/30]. \quad (56)$$

Very small values of $2\phi_1R_0$ correspond to small bubbles or the isothermal case, and again dissipation arising from heat conduction vanishes. However, there is a transition region between small and large values of $2\phi_1R_0$ where the thermal dissipation is a maximum. In this transition region, the relation between the pressure and the total gas volume can be expressed as $P_1'V_1^{\eta}$ equals a constant, where the exponent η varies from unity to γ; the state is polytropic. Figure 1, a plot of the thermal damping constant $\omega b_{th}/k$ vs $2\phi_1R_0$, clearly illustrates this transition region between the thermal and adiabatic states.

The stiffness of gas bubbles is also important since the resonant frequency for the bubble system is directly proportional to the square root of the restoring stiffness. By comparing the real terms on both sides of Eq. (53), the stiffness is expressed as

$$\frac{\gamma P_0}{kV_0}=\left[1+\left(\frac{\omega b_{th}}{k}\right)^2\right]$$

$$\times\left[1+\frac{3(\gamma-1)}{2\phi_1R_0}\left(\frac{\sinh(2\phi_1R_0)-\sin(2\phi_1R_0)}{\cosh(2\phi_1R_0)-\cos(2\phi_1R_0)}\right)\right].$$

For large values of $2\phi_1R_0$, i.e., for large bubbles and the stiffness approaching the adiabatic stiffness, the dimensionless stiffness $\gamma P_0/kV_0$ is given to within one percent by

$$\frac{\gamma P_0}{k_{ad}V_0}\simeq\left[1+\frac{3(\gamma-1)}{2\phi_1R_0}\left(1+\frac{3(\gamma-1)}{2\phi_1R_0}\right)\right].$$

Pfriem, due to an oversight, used only the first power of $\omega b_{th}/k$ in obtaining Eq. (58), and, therefore, derived an erroneous result. The stiffness, as given by the foregoing equation, is used to determine the thermal damping constant at resonance. Consequently, the thermal damping constant at resonance obtained by Pfriem is incorrect. For small values of $2\phi_1R_0$, i.e., for small bubbles and the stiffness approaching the isothermal stiffness, the dimensionless stiffness is given to within one percent by

$$\frac{\gamma P_0}{k_{iso}V_0}\simeq\left[\gamma-\frac{(2\phi_1R_0)^4}{1890}\left(1-\frac{2.1(\gamma-1)^2}{\gamma}\right)\right].$$

When the dimensionless stiffness $kV_0/\gamma P_0$ is plotted as a function of $2\phi_1R_0$, as in Fig. 2, the dimensionless stiffness approaches γ^{-1} for small values of $2\phi_1R_0$ ...

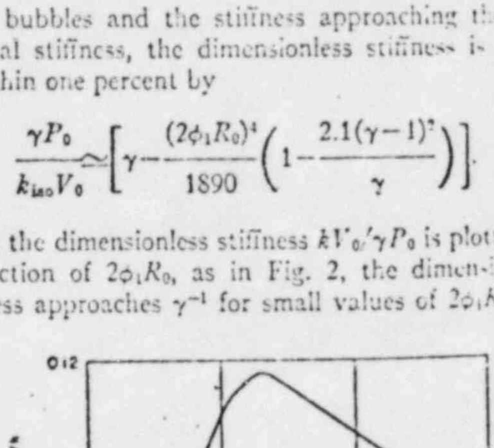

FIG. 1. Thermal damping constant vs dimensionless parameter $2\phi_1R_0$.

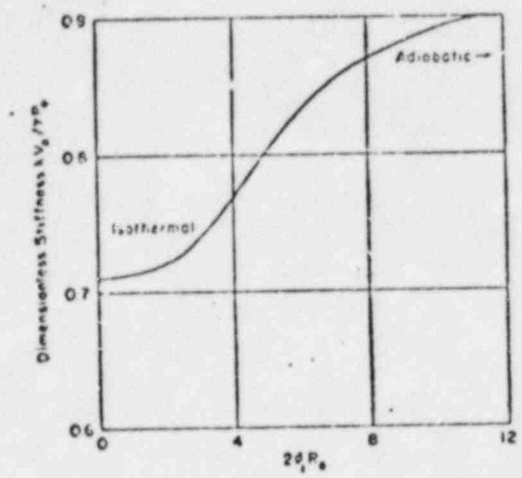

FIG. 2. Dimensionless stiffness vs dimensionless parameter $2\phi_1 R_0$.

proaches unity for large values of $2\phi_1 R_0$. The restoring stiffness of the gas and the thermal damping constant are now known. The only remaining task is to introduce the correct expression for the stiffness into the equation for the frequency, and then determine the thermal damping constant δ_{th} at resonance.

The correct expression for the stiffness will now be introduced into the equation for the resonant frequency. Minnaert derived the equation for the resonant frequency of pulsating gas bubbles in liquids by considering an adiabatic equation of state. However, the state is polytropic; the stiffness constant k for large bubbles, which is of practical importance in discussing underwater sound transmission, can be obtained from Eq. (58):

$$k=\gamma P_0/V_0\alpha, \tag{60}$$

where

$$\alpha=\left[1+\frac{3(\gamma-1)}{2\phi_1 R_0}\left(1+\frac{3(\gamma-1)}{2\phi_1 R_0}\right)\right].$$

The factor α describes the departure of the bubble stiffness from the adiabatic stiffness. Consequently, Eq. (1) becomes

$$f=(3\gamma P_0/\rho_2\alpha)^{\frac{1}{2}}/2\pi R_0=f_M/\alpha^{\frac{1}{2}}. \tag{61}$$

In the discussion so far, the instantaneous pressure inside the bubble has been considered the same as the instantaneous pressure on the surface of the bubble. However, when the bubble is small, the surface tension pressure increases the pressure inside the bubble; consequently, the instantaneous pressure inside the bubble is greater than the instantaneous pressure on the bubble surface. Smith,[14] Briggs, Johnson, and Mason,[15] Spitzer,[12] and Robinson and Buchanan[16] are among some of the investigators who have discussed the effect of surface tension on the bubble stiffness. The problem is to relate the pressure on the surface of the bubble, which is associated with the change in bubble volume, to the pressure inside the bubble. The stiffness k' is defined as

$$k'=-(\partial P_2'/\partial V_1). \tag{18b}$$

The pressure P_i inside the bubble is

$$P_i=P_2'+(2\sigma/R_1), \tag{62}$$

where P_2' is the pressure on the bubble surface, σ is the surface tension, and R_1 is the instantaneous bubble radius. The polytropic equation of state for the gas inside the bubble is

$$P_i=[P_0+(2\sigma/R_0)][V_0/V_1]^\eta. \tag{63}$$

Accordingly,

$$P_2'=P_i-2\sigma/R_1=[P_0+(2\sigma/R_0)](V_0/V_1)^\eta-2\sigma/R_1 \tag{64}$$

and, as η is γ/α, then

$$k'=(\eta P_0/V_0)[1+(2\sigma/P_0R_0)-(2\sigma/3\eta P_0R_0)] \tag{65a}$$

$$k'=\eta P_0 g/V_0=\gamma P_0 g/V_0\alpha, \tag{65b}$$

where

$$g=1+(2\sigma/P_0R_0)-(2\sigma/3\eta P_0R_0).$$

Therefore, the correct expression for the resonant frequency f_0 is

$$f_0=(3\gamma P_0 g/\rho_2\alpha)^{\frac{1}{2}}/2\pi R_0=f_M(g/\alpha)^{\frac{1}{2}}. \tag{66}$$

When the bubbles are very large, the stiffness is the adiabatic stiffness and also surface tension effects are negligible; consequently, the ratio g/α is unity and the resonant frequency is given exactly by Minnaert's equation.

Since damping is of prime importance at resonance, the thermal damping constant will now be determined as a function of the resonant frequency. At resonance, the parameter $\phi_1 R_0$ is

$$\phi_1 R_0=R_0(\omega_0/2D_1)^{\frac{1}{2}}=(3\gamma P_0 g/4\pi\rho_2 D_1 f_0\alpha)^{\frac{1}{2}}=(Fg/f_0\alpha)^{\frac{1}{2}}, \tag{67}$$

where F is $(3\gamma P_0/4\pi\rho_2 D_1)$. For a given pressure, the characteristic frequency F is a constant for the gas. When Eq. (67) is squared and the expression for α substituted, a quadratic equation for $\phi_1 R_0$ results in terms of the resonant frequency f_0, characteristic frequency F, and g. By substituting this value of $\phi_1 R_0$ into Eq. (55), the thermal damping constant δ_{th} at resonance is found to be

$$\delta_{th}=\frac{\omega_0 b_{th}}{k}=2\left[\frac{\left(\frac{16}{9(\gamma-1)^2}\frac{Fg}{f_0}-3\right)^{\frac{1}{2}}-\frac{(3\gamma-1)}{3(\gamma-1)}}{\frac{16}{9(\gamma-1)^2}\frac{Fg}{f_0}-4}\right]. \tag{68}$$

[14] F. Smith, Phil. Mag. 19, 1147 (1935).
[15] Briggs, Johnson, and Mason, J. Acoust. Soc. Am. 19, 664 (1947).
[16] R. Robinson and R. Buchanan, Proc. Phys. Soc. (London) B69, 893 (1956).

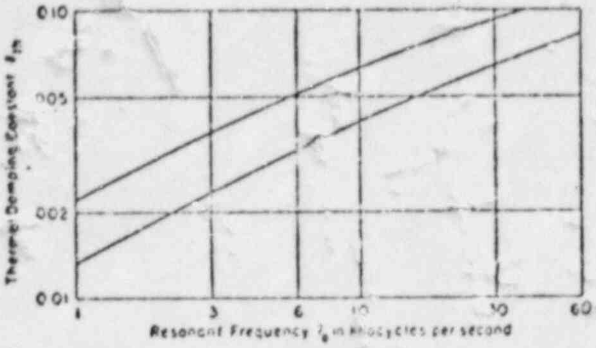

FIG. 3. Thermal damping constant for resonant air bubbles in water. The upper curve is the erroneous result obtained by Pfriem.

As Pfriem used an incorrect expression for α, he derived the following erroneous result for the resonant thermal damping constant:

$$\delta_{th}=\frac{1}{\left(\dfrac{3\gamma-1}{3(\gamma-1)}\dfrac{Fg}{f_0}\right)^{\frac{1}{2}}}\times\left[1-\frac{2}{3(\gamma-1)\left(\dfrac{3\gamma-1}{3(\gamma-1)}\dfrac{Fg}{f_0}\right)^{\frac{1}{2}}-1}\right].\qquad(69)$$

When the following values are used

$P_0=1\times10^6$ d/cm²,
$\sigma=75$ d/cm,
$K_1=5.6\times10^{-5}$ cal/cm-sec-deg c,
$\gamma=1.40$,
$s_{p1}=0.24$ cal/g,
$\rho_2=1.00$ g/cm³, and
$\rho_1=1.29\times10^{-3}$ g/cm³,

the resonant thermal damping constant δ_{th} calculated using Eq. (68) and the one calculated using Pfriem's equation can be compared. Figure 3 clearly shows that the result obtained by Pfriem is 50 to 65% too high. The resonant thermal damping constant determined in this survey agrees exactly with Willis' curve as given in the report by Spitzer. For air bubbles in water larger than 15 μ radius and with resonant frequencies less than 240 kc/sec, the thermal damping constant δ_{th} is given to within one percent by Eq. (68). When the resonant frequency is less than about 7 kc/sec, i.e., bubbles larger than 500 μ in radius, the resonant thermal constant is given to within one percent by the simple relationship:

$$\delta_{th}=[9(\gamma-1)^2/4F]^{\frac{1}{2}}(f_0)^{\frac{1}{2}}\qquad(70)$$

$$\delta_{th}=4.41\times10^{-4}f_0^{\frac{1}{2}}\ \mathrm{sec}^{\frac{1}{2}},\qquad(71)$$

disclosing that the thermal damping constant is proportional to the square root of the resonant frequency at low frequencies.

(c) Radiation Damping

In a compressible liquid, a bubble excited into volume pulsations expends a portion of its energy by radiating spherical sound waves. The bubble is considered as a simple sound source; the bubble radius R is considered small compared to the wavelength λ of the radiated sound. Smith[14] has calculated the radiation damping for gas bubbles in liquids. In order to derive the expression for the radiation damping constant, the velocity potential for a simple sinusoidal source in a compressible liquid is stated:

$$\Omega_0=-(j\omega v_1/4\pi R)\exp[j\omega(t-R/c_2)],\qquad(72)$$

where c_2 is the velocity of sound in the liquid, R is the radial distance, and v_1 is the complex amplitude of the change in bubble volume v. The change in bubble volume v is simply

$$v=v_1\exp(j\omega t),\qquad(73)$$

where the complex amplitude v_1 is defined by Eq. (47). The acoustic pressure is $\rho_2(\partial\Omega_0/\partial t)$. Consequently, at the bubble surface, the acoustic pressure is

$$p_a=-(\rho_2\omega^2v_1/4\pi R_0)[1-j(\omega R_0/c_2)-(\omega^2R_0^2/c_2^22!)+j(\omega^3R_0^3/c_2^33!)]\exp(j\omega t),\qquad(74)$$

where only the first four terms are kept in the expansion of $\exp(j\omega R_0/c_2)$. The acoustic pressure on the bubble surface is just the difference between the driving pressure on the surface of the liquid and the change in pressure on the bubble surface associated with the change in bubble volume. The change in pressure on the bubble surface is $k'v$. Therefore,

$$p_a+k'v=p_a+k'v_1\exp(j\omega t)=-P\exp(j\omega t)\qquad(75)$$

or

$$\frac{\rho_2}{4\pi R_0}\left(1-\frac{\omega^2R_0^2}{c_2^22!}\right)\ddot{v}+\frac{\rho_2\omega}{4\pi R_0}\left(\frac{\omega R_0}{c_2}-\frac{\omega^3R_0^3}{c_2^33!}\right)\dot{v}+k'v=-Pe^{j\omega t}.\qquad(76)$$

Since the term ω_0R_0/c_2 is approximately 1/75, the terms of higher order than ω_0R_0/c_2 will be neglected and Eq. (76) becomes

$$(\rho_2/4\pi R_0)\ddot{v}+(\rho_2\omega^2/4\pi c_2)\dot{v}+k'v=-P\exp(j\omega t).\qquad(77)$$

As long as the bubble radius is small compared to the wavelength of the radiated sound, it is seen that the generalized mass term for the case of a compressible liquid is the same as the corresponding term in Eq. (47) where the liquid was considered incompressible. The radiation dissipation coefficient b_{rad} is $\rho_2\omega^2/4\pi c_2$

Therefore, the resonance radiation damping constant is

$$\delta_{rad}=(b_{rad}/\omega_0 m_2)=(\omega_0 R_0/c_2)=(2\pi R_0 f_M/c_2)(g/\alpha)^{\frac{1}{2}}, \quad (78)$$

where m_2 is the generalized mass. For large bubbles, the ratio g/α is unity, and, since $2\pi R_0 f_M/c_2$ is a constant, the radiation damping constant at resonance is independent of frequency.

(d) Viscous Damping

The viscous damping constant at resonance for a pulsating spherical bubble in a viscous, incompressible liquid will now be discussed. Mallock[17] in 1910, and, later, Spitzer[12] and Poritsky[18] investigated this problem. For a pulsating bubble, the effect of viscosity is perhaps difficult to visualize. Lamb[19] states, "The only condition under which a liquid can be in motion without dissipation of energy by viscosity is that there must be nowhere any extensions or contractions of linear elements; in other words, the motion must consist of a translation and a rotation of the mass as a whole, as in the case of rigid body."

In the presence of viscosity, momentum is transmitted from one region of the liquid to another moving at a different velocity. An element of liquid moving rapidly in a particular direction tends to transmit its momentum to other elements of the liquid. The Navier-Stokes equation of motion describes the force per unit volume acting on an infinitesimal element of volume, at a point in a viscous liquid. The force per unit volume is due to the instantaneous pressure distribution of the surrounding liquid as in the case of a nonviscous liquid, and is also due to the rate of change of momentum caused by the presence of viscosity. When there are no external forces acting on each unit mass of the liquid, the equation of motion is

$$\rho[\partial/\partial t+(d\mathrm{R}/dt)\cdot\nabla](d\mathrm{R}/dt)$$
$$=-\nabla p_0+(\mu/3)\{\nabla[\nabla\cdot(d\mathrm{R}/dt)]\}+\mu\nabla^2(d\mathrm{R}/dt), \quad (79)$$

where $d\mathrm{R}/dt$ is the radial velocity vector, p_0 is the mean pressure, and μ is the coefficient of viscosity. As the liquid is considered to be incompressible, the divergence of the velocity vanishes so that the second term on the right side of the equation is zero. The remaining viscous term $\mu\nabla^2(d\mathrm{R}/dt)$ is also zero since the motion of the liquid is irrotational and the velocity can be expressed as the gradient of a scalar velocity potential. Therefore, there are no net viscous forces acting inside the liquid for the case of a pulsating spherical bubble in an incompressible, viscous liquid. Due to the presence of viscosity, momentum is transmitted through the liquid, but each infinitesimal element of liquid volume receives just as much as it gives. Therefore, there is no net viscous force acting on any element of volume internal to the liquid. The Navier-Stokes equation is not applicable for discussing the effect of viscosity for the case of a pulsating spherical bubble.

However, even though the net viscous forces in the liquid vanish, there are viscous forces acting at the surface of the bubble where they exert an excess pressure. Mallock gives us a physical picture of the effect of viscosity on a pulsating bubble by considering a small element of a spherical shell of liquid at the bubble surface. This element has definite radial and lateral dimensions at the instant the bubble radius is at its mean position. When the bubble expands, the small liquid element is distorted; the radial thickness decreases while the lateral dimension increases. Likewise, when the bubble contracts, the liquid element is again distorted; this time the radial thickness increases and the lateral dimension decreases. Since the liquid is incompressible, the distortion is not caused by a change in the volume of the liquid element but by viscous stresses. Consequently, more energy is required to compress the bubble than is regained in the subsequent expansion. Due to the radial motion and spherical symmetry, the principal directions of stress and rate of strain must be radial. Accordingly, the net stress dyadic S_R is[20]

$$S_R=-(2\mu/3)\nabla\cdot(d\mathrm{R}/dt)I+2\mu X_R, \quad (80)$$

where I is the idem factor and X is the rate of pure strain dyadic. Since the rate of pure strain is the gradient of the radial velocity and the liquid is incompressible, the radial stress at the surface of the bubble is

$$S_R=2\mu\nabla(d\mathrm{R}/dt)=-(\mu\dot{v}/\pi R_0^3). \quad (81)$$

When the effect of viscosity is included, the equation of motion for the bubble system is

$$(\rho_2/4\pi R_0)\ddot{v}+(\mu/\pi R_0^3)\dot{v}+k'v=-P\exp(j\omega t), \quad (82)$$

where $(\mu/\pi R_0^3)$ is the viscous dissipation coefficient b_{vis}. Therefore, the viscous damping constant at resonance is

$$\delta_{vis}=(b_{vis}/\omega_0 m_2)=(8\pi\mu f_0\alpha/3\gamma P_0 g)$$
$$=(8\pi\mu f_M/3\gamma P_0)(\alpha/g)^{\frac{1}{2}}. \quad (83)$$

The viscous damping constant at resonance is directly proportional to the resonant frequency f_0.

(e) Total Damping Constant at Resonance

The total damping constant δ_0 at resonance is

$$\delta_0=\delta_{th}+\delta_{rad}+\delta_{vis}, \quad (84)$$

where the thermal, radiation, and viscous damping constants are given by Eqs. (68), (78), and (83),

[17] A. Mallock, Proc. Roy. Soc. (London) A84, 391 (1910).
[18] H. Poritsky, Proc. First National Congress of Appl. Mechanics, p. 813 (June 1951).
[19] See reference 4, Sec. 329.
[20] L. Page, *Introduction to Theoretical Physics* (D. Van Nostrand Company, Inc., Princeton, New Jersey, 1952), p. 274.

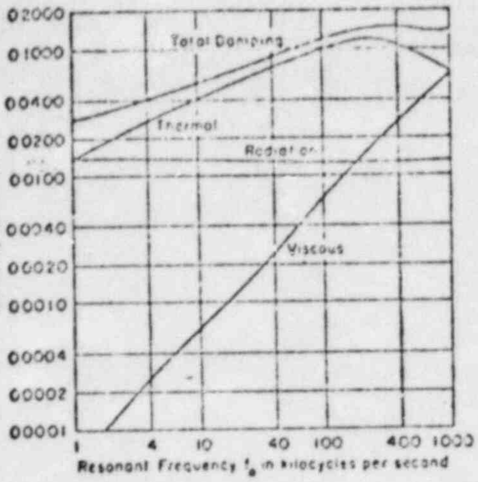

FIG. 4. Theoretical thermal, radiation, viscous, and total damping constants for resonant air bubbles in water.

respectively. Figure 4 is a plot of the thermal, radiation, viscous, and total damping constants as a function of the resonant frequency for air bubbles in water. In this figure, damping constants are given for bubbles ranging in radius from 3 μ to 3 mm.

III. EXPERIMENTAL METHODS

There are essentially four methods by which the resonant damping constant can be determined experimentally. Most of these methods are indirect ones involving the calculation of the damping from certain measured acoustical properties of the bubbles.

(a) Successive Oscillations

The method of successive oscillations is a direct process for determining the damping constant. The signal from a hydrophone, which is placed close to a pulsating bubble, is amplified and applied to the input terminals of a cathode-ray oscilloscope; the bubble pulse appears on the screen as a damped sine wave. If the amplitude of successive oscillations is plotted as a function of the cycle number of oscillation, the logarithmic decrement, and, therefore, the reciprocal Q can be determined from the slope. By knowing the time scale across the screen, the resonant frequency of the bubble system can be found.

(b) Width of the Resonance Response[21–23]

Previously, the Q for forced oscillations of a bubble was defined. Since the sound power is proportional to the square of either the radial velocity or the radial displacement, the resonant damping constant can be found by plotting the square of either of these parameters as a function of the frequency. The amplitude of oscillation of large bubbles can be found using a photoelectric method, and, for small bubbles, the radial velocity can be measured using a kind of velocity-ribbon microphone.

1. Photoelectric Method

A single gas bubble pulsating to a sonic excitation is illuminated optically and the scattered light measured by a photoelectric cell. The change in cross section of the bubble image modulates the quantity of light received at the photocell. In the photocell circuit there is generated an alternating current which is amplified and recorded on a suitable recorder. By varying the sonic excitation frequency and noting the changes in the bubble cross section, the band width and the resonant frequency can be determined.

2. Ribbon Microphone Method

A single bubble is caught on a small wax sphere fastened to a platinum thread which is placed between the poles of an electromagnet. The bubble pulsations are produced by a constant frequency magnetostrictive projector. As the bubble pulsates, the platinum thread is carried along with the oscillations and this motion of the thread produces an alternating emf which is proportional to the radial velocity. Since the generalized mass of the bubble system is considerably greater than the vibrating part of the thread mass and wax mass, the thread and wax are assumed to exert negligible influence on the resonant frequency. The bubble is allowed to grow slowly and its diameter measured with a microscope; the voltage produced by the ribbon microphone traverses a maximum as the diameter of the bubble increases. Therefore, the resonant frequency and damping constant can be determined.

(c) Standing-Wave Ratios[24–26]

A single bubble is allowed to rise freely in a liquid filled tube and pulsate under the influence of a plane progressive sound wave. The diameter of the tube is less than half a wavelength so that the sound pressure is constant over the cross section of the tube. Disturbances by reflection of this sound wave from the free surface are prevented by an absorption device or by using a pulse technique. The sound energy E_i which is radiated by a transducer at the lower end of the tube, is partly reflected E_r by the bubble, and recorded by a probe hydrophone arranged between the transducer and the bubble. The resonant damping constant can be measured from the relative reflection coefficient $(E_r/E_i)^{\frac{1}{2}}$ of a bubble pulsating at its resonant frequency:

$$\delta_0 = (R_0/Z)(4E_r/E_i)^{\frac{1}{2}} = (R_0 p_r/Z p_i)(2)^{\frac{1}{2}}, \quad (85)$$

[21] E. Meyer and K. Tamm, Akust. Z. 4, 145 (1939).
[22] H. Lauer, Akust. Beih. No. 1, 12 (1951).
[23] Yasioka, Kawasima, and Hirano, Acustica 5, 173 (1955).
[24] M. Exner, Akust. Beih. No. 1, 25 (1951).
[25] M. Exner and W. Hampe, Acustica 3, 67 (1953).
[26] H. Haeske, Acustica 6, 266 (1956).

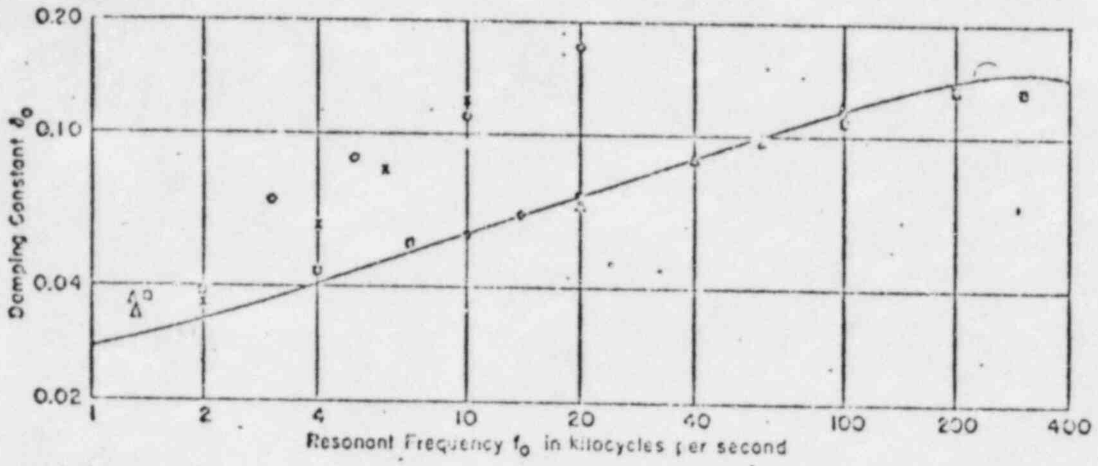

FIG. 5. The theoretical curve and experimental values for the total damping constant of resonant air bubbles in water. The points are from faired curves through the experimenter's data.

Symbol	Experimenter	Method
×	Meyer and Tamm	Width of resonance response
○	Carstensen and Foldy	Resonance absorption
Δ	Bauer	Successive oscillations
[]	Lauer	Width of resonance response
●	Exner	Standing-wave ratios
▲	Exner and Hampe	Standing-wave ratios
⊠	Haeske	Standing-wave ratios

where Z is the radius of the tube, and p_i and p_r are the sound pressures of the incident and reflected waves, respectively. The incident sound energy E_i is corrected for friction losses occurring in the tube. Therefore, the damping constant can be measured from the standing-wave ratio, and the resonant frequency is that frequency of the plane progressive sound wave which produces the maximum pulsation for the bubble.

(d) Resonance Absorption[27,28]

This method of determining the resonant damping constant depends upon measuring the attenuation of sound by a screen of bubbles. For bubbles of a single size, the attenuation through a bubble screen is a maximum at the resonant frequency of the bubbles. A projector and transmission hydrophone are located on opposite sides of the bubble screen. Each instrument is faced toward the other, and the line joining them, at the point of intersection with the bubble screen, forms an angle ϵ with the normal to the screen. In order to obtain data as to the distribution of bubbles according to size, the rate of rise of bubbles, which is a function of the bubble radius, is determined. If only a very short burst of bubbles is allowed to escape from the bubble producers and the resultant screen is observed at a height h and time t later, the screen contains only those bubbles whose rate of rise is h/t. When the bubbles are allowed to rise freely but in these definite bursts or pulses, attenuation measurements as time elapsed after the initiation of the pulse screen are made using the transmission hydrophone. This method is repeated for several projector frequencies. Carstensen and Foldy[27] give the resonant damping constant in terms of the attenuation A as

$$A=4.34n_0d\left\{\frac{[4\pi R_0{}^2\delta_0/(R')^2\delta_{rad}]}{\beta+(1\pm\beta^{\frac{1}{2}})^2\delta_0{}^2}\right\}\sec(\epsilon), \qquad (86)$$

where n_0 is the average number of bubbles per unit volume, d is the thickness of the bubble screen, R' is the off-resonant bubble radius, and β is $[(R_0/R')^2-1]^2$. In deriving this equation, Carstensen and Foldy assumed, for a bubble screen containing bubbles of essentially uniform size, that the off-resonance damping constant is $\delta_0(R_0/R')^3$.

IV. COMPARISON OF THEORY WITH EXPERIMENTAL RESULTS

The theoretical damping constant and the experimental values for the damping constants are plotted as a function of the resonant frequency in Fig. 5 indicating how well the experimental results agree with the theory of damping.

Meyer and Tamm[21] have used the width of the resonance response method to obtain the damping constant; these results are extremely high. In using the ribbon microphone, considerable damping may have been due to the oscillation of the platinum thread in the magnetic field. In addition, the experimenters themselves state the bubbles appeared dull and blurred near the resonant point; consequently, the diameters of the bubbles may not have been measured accurately with the microscope, which would affect the determina-

[27] E. Carstensen and L. Foldy, J. Acoust. Soc. Am. 19, 481 (1947).

[28] E. Meyer and E. Skudrzyk, Acustica 3, 434 (1953).

ion of the resonant frequency. The damping constant for large bubbles was determined using the photoelectric method by Meyer and Tamm, and later Lauer.[22] They used a thin wire annulus to hold the bubbles and prevent them from rising to the surface while measurements were being made. Indeed, the high damping constants found by Meyer and Tamm may be due to the wire annuli adding to the damping of the bubble system. At low frequencies, the damping constants measured by Lauer are about twenty-five percent higher than the theoretical prediction.

Bauer, formerly of the David Taylor Model Basin, used the successive oscillation method for determining the damping constant at resonance. In this experiment, the damping constant for a free bubble was measured; therefore, there is no additional damping due to a bubble holder. The bubble was formed at a nozzle; the volume pulsations started just as the bubble closed and separated from the nozzle. The unpublished damping constant measurements of Bauer are about twenty percent higher than the theory predicts.

The method of standing-wave ratios was used by Exner,[24] Exner and Hampe,[25] and Haeske[26] to determine the resonant damping constant; the results of Exner, and Exner and Hampe agree very well with the theoretical curve. According to the theory, the viscous damping becomes important around 200 kc/sec; Haeske has measured the damping constant in this frequency range. At resonant frequencies of 200-300 kc/sec, the damping constants determined by Haeske are four to eight percent lower than the theoretical curve. When the theoretical damping constant curve does not include the viscous damping constant, but only the thermal and radiation damping constants, the experimental results of Haeske are four to eight percent higher than the theoretical curve. However, the measurements by Haeske appear to be only accurate to within about ten percent. Therefore, a definite conclusion can not be formed as to whether viscous damping contributes or does not contribute to the total damping. Now, the value for the coefficient of viscosity, which enters the calculations for the viscous damping, was obtained experimentally for steady flow. At high frequencies, the value for the coefficient of viscosity may be considerably smaller than for the steady flow case. Above 40 kc/sec, Exner and Hampe very often found "anomalous" bubbles with much lower damping constants than the regular bubbles. The measured resonant frequency did not agree with the frequency calculated from the measured diameter of the bubble when Eq. (66) was used. The "anomalous" bubbles have higher frequencies than this equation predicts. There was noted, that in almost all cases the "anomalous" bubbles had dust particles on their surfaces. This increase in resonant frequency could not be explained by a decrease in the generalized mass as the dust particles would add to this mass, and there does not seem to be a logical explanation for a possible increase in the stiffness. Strasberg[27] tentatively suggested this behavior may perhaps be associated with surface oscillations of the bubble since, for very small bubbles, the frequency of surface oscillations may be of the same order as the frequency of ordinary volume pulsations. The excitation of surface oscillations by sonic excitation may require some nonsymmetry supplied by the dust particles. When Haeske performed his experiment, he took extreme care to obtain clean experimental conditions, and found no trace of "anomalous" bubbles in the 100-300 kc/sec range.

Carstensen and Foldy used the resonance absorption method to determine the resonant damping constant. The damping constant results are very high. In this method, a large number of bubbles are present, and the exact distribution in size, space, and number is difficult to determine. There may also be interaction between individual bubbles; it is difficult to state how these interactions affect the damping constant. Finally, the assumption of Carstensen and Foldy as to the behavior of the off-resonance damping constant may be invalid.

Excluding the results obtained by Meyer and Tamm and Carstensen and Foldy, the experimental damping constants agree very well with the theoretical curve. The damping constants at low frequencies obtained by Bauer and Lauer, using different experimental methods, agree quite well with each other, but their results are higher than the theory predicts. Recent work[28] indicates that for large amplitude radial pulsations, there may be some coupling between the radial pulsation and the shape oscillation. This may result in the removal of some of the energy associated with the pulsation. At high frequencies, Haeske's work does not confirm whether or not viscous damping is important.

V. SUMMARY AND CONCLUSION

Bubbles excited to volume pulsations have a polytropic equation of state for the enclosed gas which results in a phase difference between the change in pressure per unit original pressure and the change in volume per unit original volume. Therefore, the work done in compressing the bubble is more than the work done by the bubble in expanding; this difference in the work done represents a net flow of heat energy into the liquid.

Pulsating bubbles expend a portion of their energy in the form of spherical sound waves. The radiation damping is just this loss of energy.

The effect of viscosity on pulsating bubbles in an incompressible, viscous liquid is understood through the stress equations and the boundary conditions rather than the Navier-Stokes equation of motion. At the bubble surface, there are viscous forces acting

[27] M. Strasberg, Acustica 4, 450 (1954).

[28] Unpublished communication between Dr. M. Strasberg, David Taylor Model Basin, and Dr. T. B. Benjamin, King's College, Cambridge, England.

which exert an excess pressure. This results in the dissipation of energy.

Experimental results verify that the damping at resonance is due to thermal and radiation, and possibly viscous damping. The discrepancies between theory and experimental results found in measurements by Meyer and Tamm and Carstensen and Foldy are due to the particular conditions of the experiment. The small discrepancy between the theory and the results of Bauer and Lauer may be due to some coupling between the radial motion and the shape oscillations which may remove some of the energy associated with the volume pulsation.

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