Comparing Equations (3.19) and (3.20), it is concluded that the total response of the structure in terms of amplitude of transfer functions at any DOF can be calculated as the SRSS of all spatial modal solutions.

$$|H(\omega)|_k = \sqrt{\sum_{j=1}^{m} |u_{j,k}|^2} \tag{3.21}$$

It should be noted that the above derivation provides the exact solution for the linear system since there is no other assumption/simplification introduced in the solution process.

This approach is equally applicable to the calculation of structure forces and stresses due to the incoherent ground motion. Let $[K_e]$ be the element stiffness matrix. After solving Equation (3.10) and obtaining the modal displacement solutions $\{u\}_j$, the corresponding modal force solution $\{f\}_j$ can be obtained by (conversion from global coordinates to local coordinates is implied)

$$\{f\}_j = [K_e]\{u\}_j \tag{3.22}$$

Similarly, amplitude of the force/stress transfer function at any DOF k can be obtained as

$$|Q(\omega)|_k = \sqrt{\sum_{j=1}^{m} |f_{j,k}|^2} \tag{3.23}$$

In summary the above derivation confirms that, for ground motion incoherency models formulated by the PSD matrix, the SRSS summation on spatial modal solutions is an accurate method to compute the structural responses and is consistent with development of the PSD of structural response subjected to the incoherent ground motions models characterized by the PSD functions. A rational extension of this method is incorporation of the random vibration theory (RVT) in the formulation of the SSI solution. RVT approach can be directly implemented so that by providing the response spectra or PSD of input motion, the structural responses in terms of the power or response spectra can be readily computed.

Estimate of Truncation Errors

Equation (3.16) establishes the exact solution form for the auto PSD at any specified DOF. Furthermore, Equation (3.21) gives the equivalent transfer function for the DOF if the ground motions at all interaction nodes are specified by the same PSD value, S_0. A complete solution, however, is quite time consuming since it would require that the equations of motion, Equation (3.10), be solved for all m spatial modes. In engineering practice, however, only a subset of these spatial modes needs to be solved to obtain satisfactory solutions. The following section establishes an estimate on the upper bound

of potential truncation errors of the solution by using only a subset of the eigen-mode solutions.

The following characteristics of the solutions are observed from Equations (3.9), (3.11) and (3.16):

(1) For each spatial mode, the magnitude of the solution vector is directly proportional to the magnitude of the corresponding eigenvalue. In fact,

$$\{u\}_j = [H]\{u_g\}_j = [H]\sqrt{\lambda_j}\{\phi\}_j \qquad j = 1, 2, \ldots, m \tag{3.24}$$

$$|\{u_j\}|^2 = \{u_j\}^*\{u_j\} = \sum_{k=1}^{n} \overline{u_{k,j}} \cdot u_{k,j} = |\lambda_j| \cdot \| \{\phi\}_j^*[H]^*[H]\{\phi\}_j \| = |\lambda_j| \cdot C_j \tag{3.25}$$

where "*" denotes for conjugated transpose, $\| \{\phi\}_j^*[H]^*[H]\{\phi\}_j \|$ is the absolute value of the determinant of a square matrix, and C_j is that value. It can be proved that for all spatial modes, $j = 1, 2, \ldots, m$, $C_1 = C_2 = \cdots = C_m = C$ since $[H]$ is independent to the ground motions, and all $\{\phi\}_j$ are unit vectors.

(2) Following Equation (3.25), the square sum of the overall solution is directly proportional to the sum of the absolute value of the eigenvalues. i.e.,

$$\sum_{j=1}^{m} |\{u_j\}|^2 = \left(\sum_{j=1}^{m} |\lambda_j|\right) \cdot C \tag{3.26}$$

Equation (3.26) indicates that contributions of all spatial modes to the overall solutions are additive since the right-hand-side in (3.26) are positive for all j values. Thus, if we rearrange all eigenvalues from the largest to the smallest, and use a subset of only the first "s" eigen-solutions, $s << m$, the truncation error of this subset can be established as:

$$|\varepsilon|_s = \frac{1 - \sqrt{\sum_{j=1}^{s} |u_j|^2}}{\sqrt{\sum_{j=1}^{m} |u_j|^2}} = \frac{1 - \sqrt{\sum_{j=1}^{s} |\lambda_j|}}{\sqrt{\sum_{j=1}^{m} |\lambda_j|}} \qquad s << m \tag{3.27}$$

Equation (3.27) establishes the upper bound of truncation errors for the computation of the transfer functions as defined by Equation (3.21) since this estimate is for the magnitude of the entire PSD matrix and the solutions from Equation (3.21) is only a subset of the PSD solutions. This estimate is also easy to implement in practical computations since all it requires is the ratio of two sums for the eigenvalues computed in the orthogonal decomposition process of the coherency matrix.