Matrix Analysis of Electrical Machinery

SECOND EDITION

by

N. N. HANCOCK, B.SC., B.SC.(ENG.),
M.SC.TECH., PH.D., C.ENG., F.I.E.E.
SENIOR LECTURER IN ELECTRICAL ENGINEERING AT THE
UNIVERSITY OF MANCHESTER, INSTITUTE OF SCIENCE AND TECHNOLOGY

PERGAMON PRESS
OXFORD · NEW YORK · TORONTO · SYDNEY

Pergamon Press Ltd., Headington Hill Hall, Oxford
Pergamon Press Inc., Maxwell House, Fairview Park, Elmsford, New York 10523
Pergamon of Canada Ltd., 207 Queen's Quay West, Toronto 1
Pergamon Press (Aust.) Pty. Ltd., 19a Boundary Street, Rushcutters Bay, N.S.W. 2011, Australia

---

First edition 1964

Second edition 1974

Library of Congress Cataloging in Publication Data

Hancock, Norman Napoleon.
Matrix analysis of electrical machinery.
Bibliography: p.
1. Electric machinery. 2. Matrices. I. Title.
TK2211.H3 1974 621.31'04'201512943 74-3286
ISBN 0-08-017898-7
ISBN 0-08-017899-5 (flexicover)

Printed in Hungary

CHAPTER 3

Application of Matrix Algebra to Static Electrical Networks

Laplace Transform Equations

Consider the three circuits shown in Fig. 1 in which there are mutual couplings between all three coils.

The voltage equations of these circuits may be written

$$v_1 = R_1 i_1 + L_1 \frac{di_1}{dt} + \frac{1}{C_1}\int_0^t i_1 . dt + M_{12}\frac{di_2}{dt} + M_{13}\frac{di_3}{dt}$$

$$v_2 = M_{21}\frac{di_1}{dt} + R_2 i_2 + L_2\frac{di_2}{dt} + \frac{1}{C_2}\int_0^t i_2 . dt + M_{23}\frac{di_3}{dt}$$

$$v_3 = M_{31}\frac{di_1}{dt} + M_{32}\frac{di_2}{dt} + R_3 i_3 + L_3\frac{di_3}{dt} + \frac{1}{C_3}\int_0^t i_3 . dt$$

The mutual inductances of this system are all positive† with the conventional directions shown in Fig. 1.

The first of these equations can be written in operational form as

$$v_1 = (R_1 + L_1 p + 1/C_1 p) i_1 + M_{12} p i_2 + M_{13} p i_3$$

or in terms of the Laplace transforms for zero initial conditions as

$$\bar{v}_1 = (R_1 + L_1 s + 1/C_1 s)\bar{i}_1 + M_{12} s \bar{i}_2 + M_{13} s \bar{i}_3$$

The other two equations can be similarly expressed.

† See footnote on p. 37.

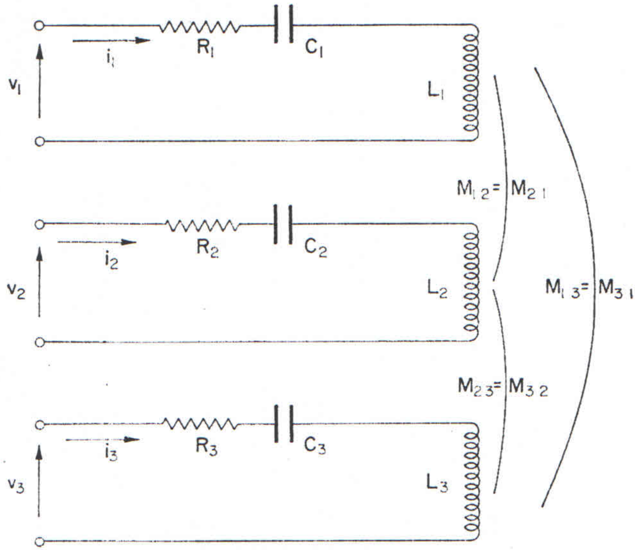

FIG. 1. Coupled circuits.

The operational equations can be abbreviated to

$$v_1 = Z_1 i_1 + X_{12} i_2 + X_{13} i_3$$
$$v_2 = X_{21} i_1 + Z_2 i_2 + X_{23} i_3$$
$$v_3 = X_{31} i_1 + X_{32} i_2 + Z_3 i_3$$

where $Z = R + Lp + 1/Cp$, and $X = Mp$ are called "transient impedances".†

The corresponding Laplace transform equations

$$\bar{v}_1 = Z_1 \bar{i}_1 + X_{12} \bar{i}_2 + X_{13} \bar{i}_3$$
$$\bar{v}_2 = X_{21} \bar{i}_1 + Z_2 \bar{i}_2 + X_{23} \bar{i}_3$$
$$\bar{v}_3 = X_{31} \bar{i}_1 + X_{32} \bar{i}_2 + Z_3 \bar{i}_3$$

where $Z = R + Ls + 1/Cs$, and $X = Ms$, form a set of linear algebraic equations with constant coefficients. To determine the currents, given

† This general use of the term "transient impedance" must be distinguished from the specific use in synchronous machine theory. See "direct-axis transient reactance", p. 235.

CHAPTER 4

Transformers

The Two-winding Transformer

The two-winding transformer consists of two coils, and if the positive directions of current are assumed to be as shown in Fig. 3, the voltage equation is

$$\begin{array}{c|c|} & \\ 1 & v_1 \\ 2 & v_2 \end{array} = \begin{array}{c|c|c|} & 1 & 2 \\ 1 & R_{11}+L_{11}p & M_{12}p \\ 2 & M_{21}p & R_{22}+L_{22}p \end{array} \begin{array}{c|c|} & \\ 1 & i^1 \\ 2 & i^2 \end{array}\dagger$$

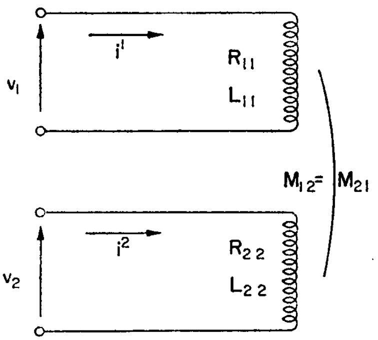

FIG. 3. Two-winding transformer.

† *The sign of mutual inductance.* A mutual inductance M_{ab}, or reactance $\omega M_{ab} = X_{ab}$, is positive when a positive rate of change of current in circuit a produces an induced e.m.f. in circuit b of the same polarity as does a positive rate of change of current in circuit b itself. In a.c. terms this means that if the currents in circuits a and b are in phase with each other, then so also are the e.m.f.s which they induce in circuit a, or those which they induce in circuit b. If the e.m.f.s are in opposition, the mutual inductance is negative.

Alternatively we may say that a mutual inductance is positive if the flux set up by positive current in one circuit links the other circuit in the same direction as the flux set up by positive current in that circuit itself.

turns. Since the test was performed with all measurements made on the primary side, it automatically yields the sum of the leakage reactances referred to the primary number of turns.

It has already been explained that it was not possible to obtain a value of x_1 with sufficient accuracy from the open-circuit test. This applies equally to x_2. In fact it is not possible by any test conducted solely at the winding terminals to get accurate values for x_1 and x_2 separately. In the absence of other information, it is usual to assume that the primary and secondary leakage reactances are equal. All parameters can then be determined from the open-circuit, short-circuit, and resistance tests.

The Three-winding Transformer

Taking the conventional directions for the three-winding transformer as shown in Fig. 6,† where all windings are treated as sinks, all voltage equations are of the same form, giving the matrix equation

$$\begin{array}{c|c|} 1 & v_1 \\ 2 & v_2 \\ 3 & v_3 \end{array} \;=\; \begin{array}{c|ccc|} & 1 & 2 & 3 \\ \hline 1 & R_{11}+L_{11}p & M_{12}p & M_{13}p \\ 2 & M_{21}p & R_{22}+L_{22}p & M_{23}p \\ 3 & M_{31}p & M_{32}p & R_{33}+L_{33}p \\ \hline \end{array} \; \begin{array}{c|c|} 1 & i^1 \\ 2 & i^2 \\ 3 & i^3 \end{array}$$

This can be referred to a common base, say the winding 1, by a transformation matrix

$$\mathbf{C} = \begin{array}{c|ccc|} & 1' & 2' & 3' \\ \hline 1 & 1 & & \\ 2 & & 1/k_2 & \\ 3 & & & 1/k_3 \\ \hline \end{array}$$

where $k_2 = N_2/N_1$ and $k_3 = N_3/N_1$, N_1, N_2, N_3 being the numbers of turns of windings 1, 2, 3 respectively.

† The relative positions of the three windings of Fig. 6 are intended to imply that M_{23} is the smallest of the three mutual inductances when all are referred to a common base.

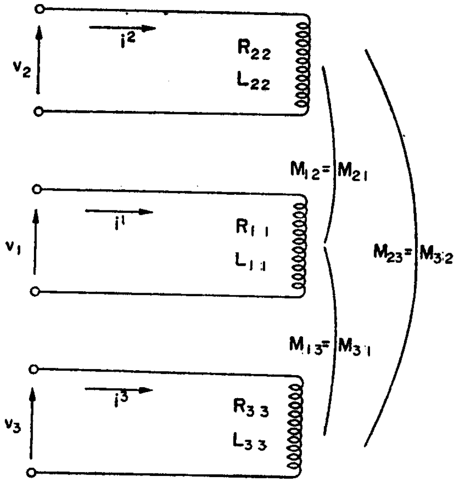

FIG. 6. Three-winding transformer.

The referred impedance matrix is

$$\mathbf{Z}' = \mathbf{C}_t\mathbf{Z}\mathbf{C} = \begin{array}{c} \\ 1' \\ 2' \\ 3' \end{array}\!\!\begin{array}{c} \begin{array}{ccc} 1 & 2 & 3 \end{array} \\ \left[\begin{array}{ccc} 1 & & \\ & 1/k_2 & \\ & & 1/k_3 \end{array}\right] \end{array} \begin{array}{c} \\ 1 \\ 2 \\ 3 \end{array}\!\!\begin{array}{c} \begin{array}{ccc} 1 & 2 & 3 \end{array} \\ \left[\begin{array}{ccc} R_{11}+L_{11}p & M_{12}p & M_{13}p \\ M_{21}p & R_{22}+L_{22}p & M_{23}p \\ M_{31}p & M_{32}p & R_{33}+L_{33}p \end{array}\right] \end{array} \begin{array}{c} \\ 1 \\ 2 \\ 3 \end{array}\!\!\begin{array}{c} \begin{array}{ccc} 1' & 2' & 3' \end{array} \\ \left[\begin{array}{ccc} 1 & & \\ & 1/k_2 & \\ & & 1/k_3 \end{array}\right] \end{array}$$

$$= \begin{array}{c} \\ 1' \\ 2' \\ 3' \end{array}\!\!\begin{array}{c} \begin{array}{ccc} 1' & 2' & 3' \end{array} \\ \left[\begin{array}{ccc} R_{11}+L_{11}p & (1/k_2)M_{12}p & (1/k_3)M_{13}p \\ (1/k_2)M_{21}p & (1/k_2)^2(R_{22}+L_{22}p) & (1/k_2k_3)M_{23}p \\ (1/k_3)M_{31}p & (1/k_2k_3)M_{32}p & (1/k_3)^2(R_{33}+L_{33}p) \end{array}\right] \end{array}$$

$$= \begin{array}{c} \\ 1' \\ 2' \\ 3' \end{array}\!\!\begin{array}{c} \begin{array}{ccc} 1' & 2' & 3' \end{array} \\ \left[\begin{array}{ccc} R'_{11}+L'_{11}p & M'_{12}p & M'_{13}p \\ M'_{21}p & R'_{22}+L'_{22}p & M'_{23}p \\ M'_{31}p & M'_{32}p & R'_{33}+L'_{33}p \end{array}\right] \end{array} \quad \text{say.}$$