

Design of Pressure Vessel Pads and Attachments To Minimize Global Stress Concentrations

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ABSTRACT

Pressure vessel pads and attachments locally stiffen the vessel shell and alter the membrane stress field due to internal pressure in the vicinity of the attachment. The problem of determining the attachment shape that produces minimum shell stresses due to internal pressure is similar to the problem in plane stress of determining the geometry of an elastic inclusion that produces minimum stress concentration in a biaxial stress field. Applying a design criterion, called the harmonic field condition, to be satisfied by the final vessel stresses in the presence of the attachment (inclusion) yields a singular integral equation in terms of the original stress field and the unknown geometry of the inclusion. The resulting solution is an ellipse of arbitrary size where the ratio of principal axes is inversely proportional to the ratio of the principal strains in the original stress field. The solution not only produces constant stress around the inclusion boundary and no shear stresses, but also no change in the volume strain energy and no elastic rotations anywhere in the stress field, which is quite remarkable. The finite element method is used to illustrate these dramatic results.

INTRODUCTION

In the design of pressure vessels it is common practice to weld a variety of attachments, such as, support pads, lifting lugs, instrument ports and reinforcements to the pressure boundary of the vessel. These attachments locally stiffen the vessel shell and alter the membrane stress field due to internal pressure in the vicinity of the attachment. Depending on the type of weld and geometry of the weld footprint, the local stress field in the vessel shell at the attachment boundary can increase significantly. While mechanical loads may create occasional high stresses at the attachment/vessel intersection, the attachment design may be dominated by pressure induced stresses and their fluctuations. In those situations where the overall stress concentration caused by the attachment must be limited, it pays to investigate alternate attachment geometries that can minimize the membrane stresses.

The problem of determining the attachment shape that produces minimum shell stresses due to internal pressure is similar to the problem in plane stress of determining the geometry of an elastic inclusion that produces minimum stress concentrations in a biaxial stress field. Determining the unknown shape of the attachment boundary (the inclusion) to satisfy specific requirements imposed on the final stress field, is an inverse, or design problem in elasticity.

In the structural design of buildings, bridges and airframes, the search for optimum structural shapes has traditionally been conducted using a strategy distinctly different from that of standard mathematical analysis where the governing equations are solved for a given set of boundary conditions on a body of specified geometry. Instead, some geometric parameters are left unspecified and extra conditions are imposed on the stress-strain field sufficient to then derive a configuration "optimized" to that preselected design criterion.

Linear and shell structures provide a classic example of this alternative design strategy in which the search for optimum shapes to eliminate bending and the resulting effects on structures has been a major historical theme. In startling contrast, there is essentially no counterpart to the design tradition in elasticity. A 200-year history has produced a rich and elegant mathematical theory in both real and complex representation, but thus far the inverse approach has generally been bypassed. There are a number of influences one might postulate to explain, at least in part, this omission, but probably the most important factor is the lack of clear design criteria for inverse elasticity that can be translated into precise mathematical statements comparable in power and scope to the no-bending or minimum energy condition in structures.

One such criterion, the harmonic field condition, has been successfully applied to the design of holes that produce absolute minimum stress concentrations [1]. The purpose of this paper is to summarize the work that has been done to apply the harmonic field condition to the design of inclusions that produce minimum stress concentrations and bring to the attention of the engineering community the benefits that can be derived when using such shapes. Since the rigid inclusion is a limiting case and approximates the behavior of stiff reinforcements and welded attachments on thin shells, it is selected to illustrate the application of the methodology.

HARMONIC FIELD CONDITION

The problem of determining the geometry of a boundary (i.e., a hole or inclusion) involves design and not simply analysis. When the geometry of a boundary is left as a variable, one must prescribe instead essential properties of the final (distorted) stress field sufficient to determine a hole or inclusion shape for that design purpose. However, one cannot arbitrarily select the local effect of a hole or inclusion on the original stress field since there may not exist a shape that can produce that effect. Thus the problem is profoundly different from the first and second fundamental problems of the theory of elasticity, where the local effect of any hole or inclusion on the stress field is uniquely determined by its known geometry. As such, in any inverse strategy the design condition to be satisfied by the final stress field is absolutely crucial.

Since boundary conditions uniquely determine the final state of stress in the fundamental problems of elasticity, and since maximum stresses must occur on the boundary of the hole or inclusion, it would seem that a design condition requiring, for example, that the stresses around the boundary be constant, as is the case of the circular hole in an isotropic stress field, would provide sufficient conditions for a solution and an optimum result. Actually, this approach was successfully implemented by Cherepanov [2] in the design of a hole in a biaxial stress field. [Cherepanov's work was unknown to the author at the time the author's research was being conducted.] The author initially considered the constant stress approach, but rejected it, believing it to be the manifestation of a more powerful design condition that would incorporate the fundamental properties of the final stress field everywhere, not just on the hole boundary [1]. In fact, this design requirement (constant boundary stress) is obtained as a special case of the "harmonic field condition" described herein where both the stress field and boundary loads are prescribed as constant. As such, the constant boundary stress requirement is severely limited, and, in fact, there is no hole shape that can satisfy this design requirement if the original stress field is not constant [3].

It is well known that in an isotropic stress field the circular shape for holes or rigid inclusions produces constant stress around the boundary and minimum stress concentrations. Less known is the fact that the first invariant (i.e., the sum of the normal stresses, $I_1 = \sigma_1 + \sigma_2 = \sigma_x + \sigma_y$) is not only constant on the boundary, but is also constant everywhere in the final (distorted) stress field and equal in magnitude to the first invariant in the original (undistorted) stress field prior to the introduction of the circular hole or inclusion. With the circular shape as a model, it was decided to choose for the design condition: *That the first invariant of the stress (or strain) remain unchanged everywhere in the field when the hole or inclusion is introduced.*

This design condition has broad implications, since the first invariant satisfies Laplace's equation. Thus the design condition can be restated such: *That the Laplacian component of the stress field remains unchanged everywhere when the hole or inclusion is introduced.* Since functions that satisfy Laplace's equation are analytic and are called harmonic functions, the term "harmonic field condition" is used to describe the design condition, and the term "harmonic shapes" is used to describe the geometry of these special holes and inclusions.

Simply stated: *Harmonic shapes do not perturb the Laplacian component of the original stress field.* As a consequence, harmonic shapes introduce only pure shear (or only a change in distortion strain energy), no change in volume strain energy, and no elastic rotation. [1], [4]. Thus, the analogy to membrane shapes for arches and shells is complete. For such structures, the no bending condition is actually a requirement that there be no elastic rotation of any cross section, and the harmonic field condition gives a full two-dimensional counterpart imposing "no bending" anywhere in the field when the harmonic shape is introduced.

HARMONIC SHAPES

General Solution

Since we seek the geometry of an unknown boundary, a complex-variable formulation, such as presented by Muskhelishvili [5] offers a particularly powerful analytic approach. An opening of essentially any shape can be described as a function of a complex variable $z = x + iy$, and it becomes convenient to map from a region containing a circular boundary using the inverse transformation $z = \omega(\zeta)$. Letting $\zeta = \rho e^{i\theta}$, the polar coordinates ρ , θ may be considered the curvilinear coordinates of a point x , y of the z -plane related by the equation $z = \omega(\zeta)$. Stresses, strains and displacements for the plane problem without body forces can be expressed in terms of two complex functions $\phi(\zeta)$ and $\psi(\zeta)$, which correspond to the final stress field. These complex functions can, in turn, be represented in the form [6]:

$$\phi(\zeta) = \phi_o(\zeta) + \phi^*(\zeta) \quad (1)$$

$$\psi(\zeta) = \psi_o(\zeta) + \psi^*(\zeta) \quad (2)$$

where $\phi_o(\zeta)$ and $\psi_o(\zeta)$ are known functions of the original stress field and $\phi^*(\zeta)$ and $\psi^*(\zeta)$ are unknown functions that account for the perturbation of the stress field due to the presents of the hole or inclusion. In the standard problems of elasticity the unknown functions $\phi^*(\zeta)$ and $\psi^*(\zeta)$ are determined from the boundary condition:

$$\phi(\sigma) + \frac{\omega(\sigma)\overline{\phi'(\sigma)}}{\overline{\omega'(\sigma)}} + \overline{\psi(\sigma)} = F(\sigma) \quad (3)$$

or

$$\kappa\phi(\sigma) - \frac{\omega(\sigma)\overline{\phi'(\sigma)}}{\overline{\omega'(\sigma)}} - \overline{\psi(\sigma)} = 2\mu(g_1 + ig_2) = D(\sigma) \quad (4)$$

in which σ (without a subscript) equals the value of ζ on the boundary of the unit circle γ (i.e., $\sigma = e^{i\theta}$). Eq. 3 is for the first fundamental problem of elasticity in which $F(\sigma)$ represents any self-equilibrating boundary loading, while Eq. 4 is for the second fundamental problem in which g_1 and g_2 are the known boundary values of the displacement components u and v imposed on γ . The material properties κ and 2μ may also be written for plane stress as:

$$2\mu = \frac{E}{(1+\nu)}; \quad \kappa = \frac{3-\nu}{1+\nu} \quad (5)$$

and for plane strain as:

$$2\mu = \frac{E}{(1+\nu)}; \quad \kappa = 3-4\nu \quad (6)$$

in terms of Young's modulus, E , and Poisson's ratio ν .

As previously mentioned, for standard analysis where geometry is known, the problem becomes the determination of the unknown complex functions ϕ^* and ψ^* . For the inverse problem, on the other hand, the desired shape will be given directly by the unknown mapping function $\omega(\sigma)$ which can be found from the applicable boundary condition only if the unknown perturbation components ϕ^* and ψ^* can be eliminated. Such is the case since it can be shown [1] that the harmonic field condition simply requires that:

$$\phi^*(\zeta) = 0 \quad (7)$$

and the Cauchy-type singular integral

$$\int_{\gamma} \frac{\overline{\psi^*(\sigma)}}{\sigma - \zeta} d\sigma = 0 \quad (8)$$

since $\psi^*(\sigma)$ is the boundary value of a function holomorphic outside γ . Thus, multiplying each term in Eq.3 and Eq.4 by $d\sigma/(\sigma-\zeta)$ and integrating over the boundary, the basic equations for harmonic shapes are found to be:

$$\int_{\gamma} \frac{\phi_o(\sigma)}{\sigma - \zeta} d\sigma + \int_{\gamma} \frac{\omega(\sigma)\overline{\phi_o'(\sigma)}}{\overline{\omega'(\sigma)}(\sigma - \zeta)} d\sigma + \int_{\gamma} \frac{\overline{\psi_o(\sigma)}}{\sigma - \zeta} d\sigma = \int_{\gamma} \frac{F(\sigma)}{\sigma - \zeta} d\sigma \quad (9)$$

or

$$\int_{\gamma} \frac{\kappa\phi_o(\sigma)}{\sigma - \zeta} d\sigma - \int_{\gamma} \frac{\omega(\sigma)\overline{\phi_o'(\sigma)}}{\overline{\omega'(\sigma)}(\sigma - \zeta)} d\sigma - \int_{\gamma} \frac{\overline{\psi_o(\sigma)}}{\sigma - \zeta} d\sigma = \int_{\gamma} \frac{D(\sigma)}{\sigma - \zeta} d\sigma \quad (10)$$

Eq.9 and Eq.10 determine the geometry ($\omega(\sigma)$) of the hole or inclusion in terms of the known stress field ($\phi_o(\sigma)$, $\psi_o(\sigma)$) and the forces ($F(\sigma)$) or displacements ($D(\sigma)$) imposed on the boundary. At first glance, these expressions, which may involve nonlinear singular integrals, are not particularly enticing. However, as demonstrated below, they are, in fact, manageable at least for the free fields approximated in most problems found in practice.

The basic equations are derived with general boundary loading and displacement functions $F(\sigma)$ and $D(\sigma)$, and the significance of this capability to control the harmonic shape for particular applications must be emphasized. Practically any combination of internal tractions and displacements expressed as a function of an angular coordinate, θ , could be examined as, for example, prestress with expandable liners or "shrink fitting."

HARMONIC HOLES

Consider a biaxial stress field with principal stresses N_1 , N_2 which have constant magnitude and direction at every point in the plane such that N_1 coincides with the x -axis and $|N_1| \geq |N_2|$. In addition a uniformly distributed internal pressure of magnitude P is also applied to the unknown hole boundary. For these conditions the harmonic hole is an ellipse with major axis "a" oriented parallel to the direction of the major principal stress N_1 and where the ratio of the major to minor axes of the ellipse is [1]:

$$\frac{a}{b} = \frac{N_1 - P}{N_2 - P} \quad (11)$$

In the absence of internal pressure the ratio becomes:

$$\frac{a}{b} = \frac{N_1}{N_2} \quad (12)$$

The resulting stress is constant on the boundary and equal to:

$$\sigma_0 = N_1 + N_2 \quad (13)$$

which is the absolute minimum stress possible for any hole shape in a biaxial field [1].

Figure 1 shows the proportions of the harmonic hole in a biaxial field where $N_1 = 2N_2$. [A number of aircraft manufacturers use these proportions in the design of airplane windows.] Harmonic holes for nonconstant fields are given in Reference 3. Figures 2 and 3 are plots of the principal tensile stresses around a circular hole and the harmonic hole from a finite element solution for the case of a biaxial stress field where $N_1 = 69.0$ Mpa (10,000psi) and $N_2 = 34.5$ Mpa (5,000psi). Note that the stress concentration factor for the circular hole is 2.5, whereas for the harmonic hole it is only 1.5, a significant reduction.

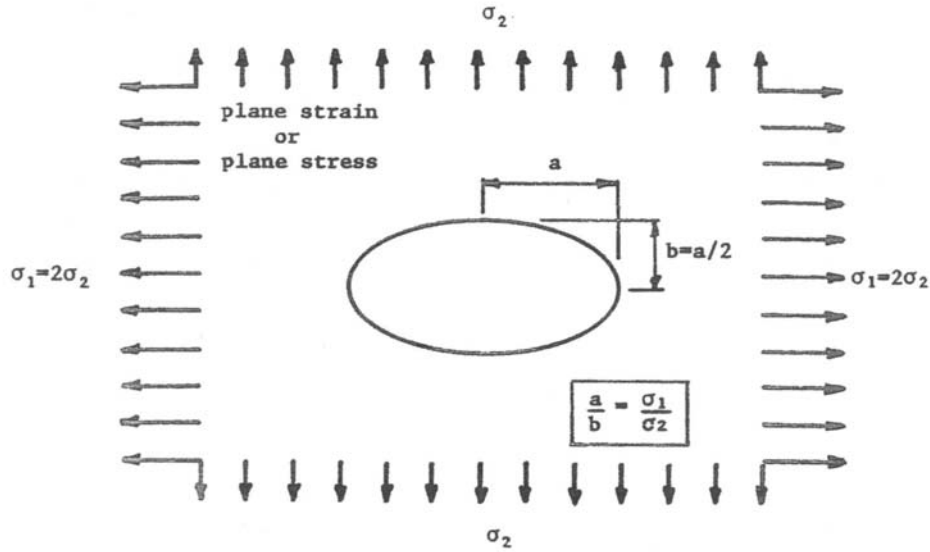


Figure 1: Harmonic Hole in a Biaxial Stress Field

RIGID HARMONIC INCLUSION

Again, consider a biaxial stress field with principal stresses N_1, N_2 which have constant magnitude and direction at every point in the plane such that N_1 coincides with the x-axis and $|N_1| \geq |N_2|$. For these conditions [5]:

$$\phi_o(\zeta) = \Gamma \omega(\zeta); \quad \psi_o(\zeta) = \Gamma' \omega(\zeta); \quad \phi_o'(\zeta) = \Gamma \omega'(\zeta) \quad (14)$$

in which

$$\Gamma = \bar{\Gamma} = \frac{1}{4}(N_1 + N_2); \quad \Gamma' = \bar{\Gamma}' = -\frac{1}{2}(N_1 - N_2) \quad (15)$$

Substituting into the general integral Eq.10 and combining similar terms, one obtains

$$\Gamma(\kappa - 1) \int_{\gamma} \frac{\omega(\sigma) d\sigma}{\sigma - \zeta} - \Gamma' \int_{\gamma} \frac{\overline{\omega(\sigma)} d\sigma}{\sigma - \zeta} = 0 \quad (16)$$

as the functional equation to be solved for $\omega(\sigma)$. Letting $\omega(\sigma)$ have the form

$$\omega(\zeta) = R \left[\zeta + \frac{a_1}{\zeta} + \frac{a_2}{\zeta^2} + \dots \right] \quad (17)$$

the integral can be evaluated to give

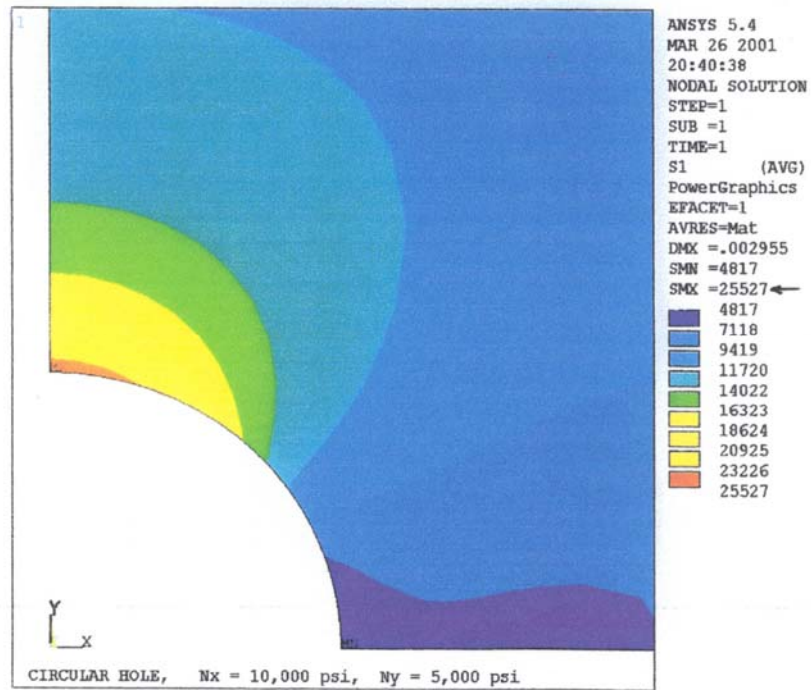


Figure 2: Principal Tensile Stress Contours for a Circular Hole in a Biaxial Stress Field where $N_x = 69.0$ MPa (10,000 psi) and $N_y = 34.5$ Mpa (5,000 psi)

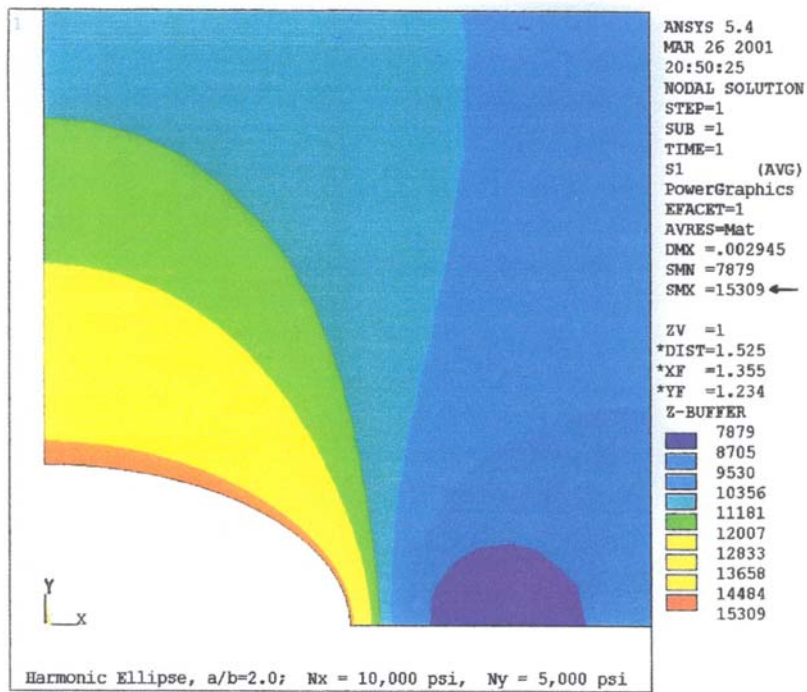


Figure 3: Principal Tensile Stress Contours for the Harmonic Hole in a Biaxial Stress Field where $N_x = 69.0$ MPa (10,000 psi) and $N_y = 34.5$ Mpa (5,000 psi)

$$-\Gamma(\kappa-1)R\left[\frac{a_1}{\zeta} + \frac{a_2}{\zeta^2} + \dots\right] + \Gamma'R\left[\frac{1}{\zeta}\right] = 0 \quad (18)$$

in which the “a” quantities are constants and R is a real scaling constant. Equating similar powers of ζ , one obtains:

$$a_1 = a_2 = \dots = 0; \quad a_1 = \frac{\Gamma'}{\Gamma(\kappa-1)} \quad (19)$$

Thus the rigid harmonic inclusion is an ellipse of arbitrary size with the ratio of major to minor axes:

$$\frac{a}{b} = \frac{(\kappa-3)N_1 + (\kappa+1)N_2}{(\kappa+1)N_1 + (\kappa-3)N_2} \quad (20)$$

or, in terms of principal normal strains, simply

$$\frac{a}{b} = \frac{\varepsilon_2}{\varepsilon_1} \quad (21)$$

where “a” coincides with the x-axis. In an isotropic field where $N_1 = N_2$ the shape, as expected is a circle. Figure 4 shows the proportions of the rigid harmonic inclusion in a biaxial stress field for the case of plane stress where $N_1 = 2N_2$.

The stresses on the boundary of any rigid inclusion are [6]:

$$\sigma_\rho = (1+\kappa) \operatorname{Re} \left[\frac{\phi'(\sigma)}{\omega'(\sigma)} \right]; \quad \sigma_\theta = (3-\kappa) \operatorname{Re} \left[\frac{\phi'(\sigma)}{\omega'(\sigma)} \right]; \quad \tau_{\rho\theta} = (1+\kappa) \operatorname{Im} \left[\frac{\phi'(\sigma)}{\omega'(\sigma)} \right]. \quad (22)$$

However, because of the harmonic field condition, $\phi^* = 0$

$$\sigma_\rho = \frac{(\kappa+1)}{4}(N_1 + N_2); \quad \sigma_\theta = \frac{(3-\kappa)}{4}(N_1 + N_2); \quad \tau_{\rho\theta} = 0. \quad (23)$$

As can be seen, harmonic inclusions not only give the required condition

$$\sigma_\rho + \sigma_\theta = N_1 + N_2 \quad (24)$$

at the interface, but also no shears, so that in fact, the contact stresses are principal with no tendency for slip to occur at the interface (See Figure 4). Figure 5 is a plot of the principal tensile stress and the octahedral shear stress around the rigid harmonic inclusion from a plane stress finite element solution where $N_1 = N_x = 69.0$ Mpa (10,000psi), $N_2 = N_y = 34.5$ Mpa (5,000psi) and Poisson's ratio equals 0.25. [The reader should be aware of the very small error in the finite element solution when compared to the exact solution.]

PROOF OF MINIMUM STRESS CONCENTRATION

It is clear that harmonic holes must produce minimum stress concentrations in constant fields [1], since maximums and minimums in Laplacian fields must occur on the boundary. It can also be shown that the same is true for a rigid harmonic inclusion. For example, for plane stress the in-plane normal strains are given by;

$$\varepsilon_\theta = \frac{1}{E}(\sigma_\theta - \nu\sigma_\rho); \quad \varepsilon_\rho = \frac{1}{E}(\sigma_\rho - \nu\sigma_\theta). \quad (25)$$

Thus at the boundary of *any* rigid inclusion:

$$\varepsilon_\theta = 0; \quad \sigma_\theta = \nu\sigma_\rho. \quad (26)$$

and therefore:

$$\sigma_\rho + \sigma_\theta = (1+\nu)\sigma_\rho = \frac{1+\nu}{\nu}\sigma_\theta \quad (27)$$

But the minimum value possible for the first invariant in a constant field is simply $N_1 + N_2$ so the minimum stresses possible on the boundary are:

$$\sigma_\rho = \frac{N_1 + N_2}{1+\nu}; \quad \sigma_\theta = \frac{\nu}{1+\nu}(N_1 + N_2). \quad (28)$$

which are, in fact, those for the harmonic inclusion Eq.23. The proof for plane strain is similar.

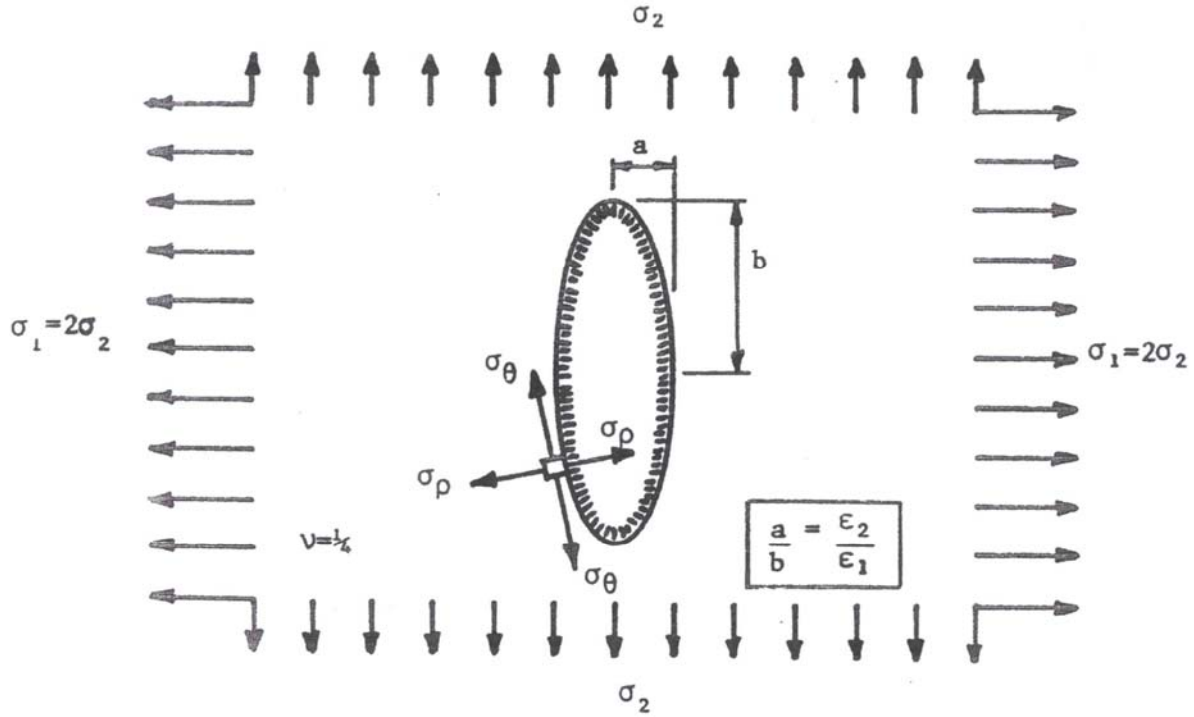


Figure 4: The Rigid Harmonic Inclusion for a Biaxial Stress Field

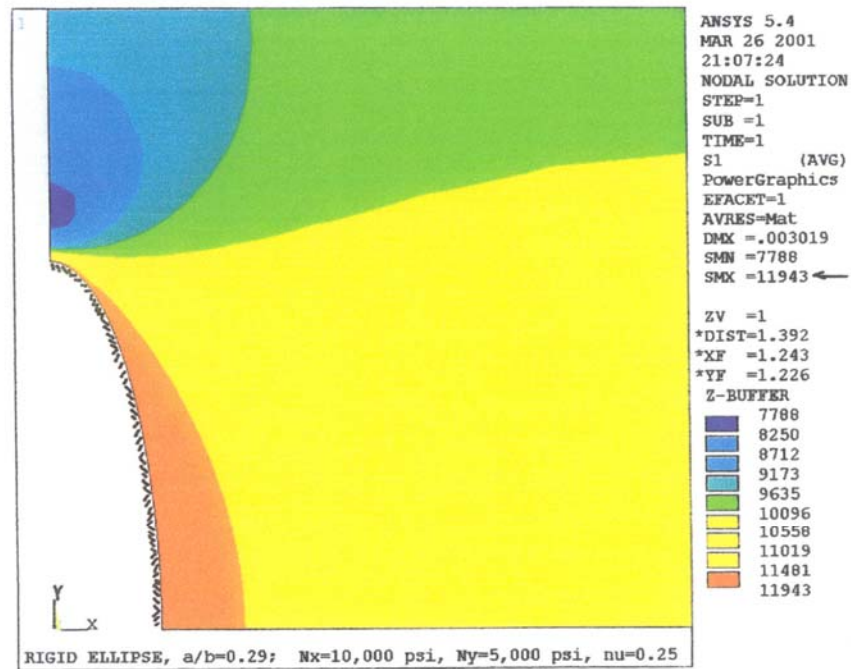


Figure 5: Principal Tensile Stress Contours for a Rigid Harmonic Inclusion in a Biaxial Stress Field where $N_x = 69.0$ MPa (10,000 psi), $N_y = 34.5$ Mpa (5,000 psi) and Poisson's Ratio = 0.25

CONCLUSION

Application of the harmonic field condition to the design of inclusions in a biaxial stress field results in a rigid inclusion (attachment) shape in the form of an ellipse where the ratio of the major to minor axes is inversely proportional to the principal strains in the original stress field, or:

$$\frac{a}{b} = \frac{\varepsilon_2}{\varepsilon_1} \quad (29)$$

This simply result not only produces the absolute minimum stress concentration for any inclusion shape, but also no shear stress on the inclusion boundary, so that, in fact, the contact stresses are principal. In addition, the rigid harmonic inclusion produces no change in volume strain energy from the original field and no elastic rotations anywhere in the stress field.

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